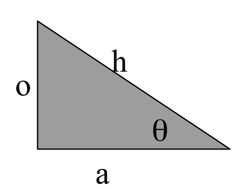
CMSC 435
Introductory Computer Graphics
Math Review

Penny Rheingans

UMBC

Basic Trigonometry

- Angles
  - degrees = $180/\pi$ radians
- Some functions
  - $\sin\theta = o/h$
  - $\cos\theta = a/h$
  - $\tan\theta = o/a$

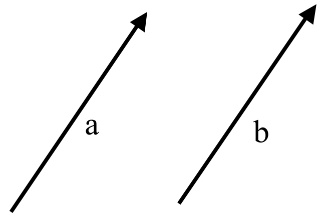


- A few identities
  - $\sin(-A) = -\sin A$
  - $\cos(-A) - \cos A$
  - $\sin^2 A + \cos^2 A = 1$

Vectors

- Vector: length and direction gives offset

$$a = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \quad a^T = \begin{bmatrix} x_a & y_a & z_a \end{bmatrix}$$

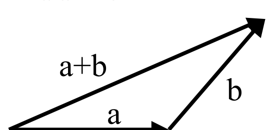


Vectors (2)

- Operations
  - Length

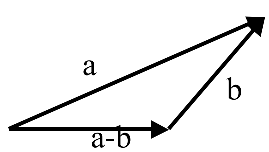
$$\|a\| = \sqrt{x_a^2 + y_a^2 + z_a^2}$$

  - Addition



$$a + b = \begin{bmatrix} x_a + x_b \\ y_a + y_b \\ z_a + z_b \end{bmatrix}$$

  - Subtraction



$$a - b = \begin{bmatrix} x_a - x_b \\ y_a - y_b \\ z_a - z_b \end{bmatrix}$$

Vectors (3)

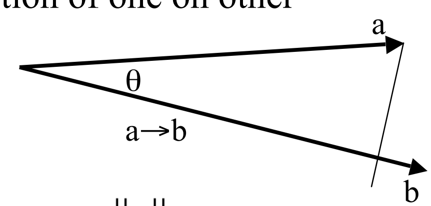
- Dot Product

$$a \cdot b = x_a x_b + y_a y_b + z_a z_b$$

  - angle between

$$a \cdot b = \|a\| \|b\| \cos\theta$$

  - projection of one on other



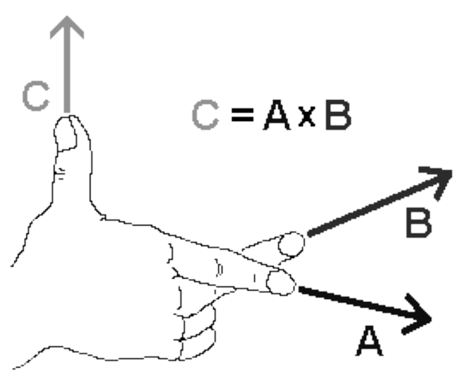
$$a \rightarrow b = \|a\| \cos\theta$$

Vectors (4)

- Cross Product

$$a \times b = (y_a z_b - z_a y_b, z_a x_b - x_a z_b, x_a y_b - y_a x_b)$$

- c= a×b
  - orthogonal to a,b
  - length=pgram area
- b×a = -a×b



Vectors (5)

- Coordinate Frames (uvw coord system)
  - Orthonormal basis
    - Unit length $\|u\| = \|v\| = \|w\| = 1$
    - Orthogonal $u \cdot v = v \cdot w = w \cdot u = 0$
  - Right-handed vs left-handed
    - Right-handed $w = u \times v$
- Coordinate frames
  - Global (world) coordinate system
  - Local (object) coordinate system

Orthonormal Basis

- Constructing from a vector **a**
  - Unit vector in direction of **a**: $w = \frac{a}{\|a\|}$
  - Any perpendicular to **w** (using noncollinear **t**): $u = \frac{t \times w}{\|t \times w\|}$
  - Unit vector perpendicular to both: $v = w \times u$
- Constructing from two vectors **a**, **b**
  - Unit vector in direction of **a**: $w = \frac{a}{\|a\|}$
  - Perpendicular to w and b: $u = \frac{b \times w}{\|b \times w\|}$
  - Unit vector perpendicular to both: $v = w \times u$

Linear Interpolation

- Formula

$$n = n_1 + t(n_2 - n_1)$$

where $n=n_1$ at $t=0$, $n=n_2$ at $t=1$

- Uses
  - Points along a line (repeat for x,y,z)
  - Colors between sample points (repeat for r,g,b)

Implicit Lines

- Function defines relationship among coords
- 2D Formula

$$y - mx - b = 0$$

  - where y is slope and b is intercept
- General form

$$Ax + By + C = 0$$

  - Orthogonal to $\begin{bmatrix} A & B \end{bmatrix}$
  - Computing from points

$$\left(y_0 - y_1\right)x + \left(x_0 - x_1\right)y + x_0 y_1 - x_1 y_0 = 0$$

  - Substitute in point for signed distance

Implicit Primitives

- Function defines relationship among coords
- Implicit Plane

$$(\mathrm{p}-\mathrm{a})\cdot \mathrm{n}=0$$

  - Orthogonal to n, through a
  - Computing from points a,b,c

$$(p-a)\cdot\left((b-a)\times(c-a)\right)=0$$

- Implicit Sphere

$$(p-c)^2-r^2=0$$

  - Radius r, center point c

Parametric Lines

- Define line by end points
- Formula

$$p=p_1+t(p_2-p_1)$$

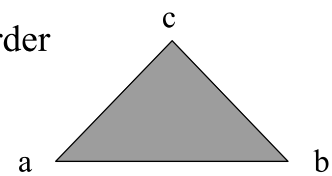
where $p=p_1$ at $t=0$, $p=p_2$ at $t=1$

- Components

$$x=x_1+t(x_2-x_1)$$
$$y=y_1+t(y_2-y_1)$$
$$z=z_1+t(z_2-z_1)$$

Triangles

- Specified by triplet of vertex positions
  - a,b,c
  - Counter-clockwise order



- Finding triangle normal

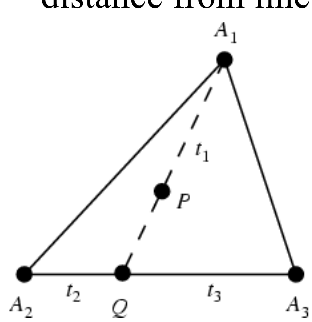
$$n = (b - a) \times (c - a)$$

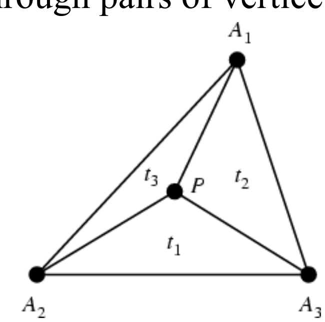
Barycentric Coordinates

- Use non-orthogonal coordinates to describe position relative to vertices

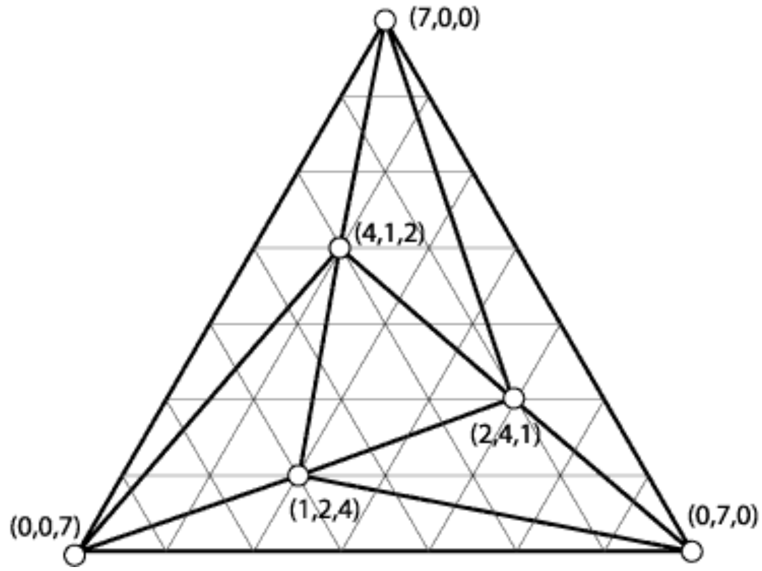
$$p = a + \beta(b - a) + \gamma(c - a) \quad p(\alpha, \beta, \gamma) = \alpha a + \beta b + \gamma c$$

  - Coordinates correspond to scaled signed distance from lines through pairs of vertices





Barycentric Example



Barycentric Coordinates

- Computing coordinates

$$\gamma = \frac{(y_a - y_b)x + (x_b - x_a)y + x_a y_b - x_b y_a}{(y_a - y_b)x_c + (x_b - x_a)y_c + x_a y_b - x_b y_a}$$

$$\beta = \frac{(y_a - y_c)x + (x_c - x_a)y + x_a y_c - x_c y_a}{(y_a - y_c)x_b + (x_c - x_a)y_b + x_a y_c - x_c y_a}$$

$$\alpha = 1 - \beta - \gamma$$

Matrix Multiplication

- With matrices A, B

$$A = \begin{bmatrix} a_{00} & a_{10} & a_{20} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{00} & b_{10} & b_{20} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{bmatrix}$$

- To compute C=AB

$$c_{ij} = a_{i0}b_{0j} + a_{i1}b_{1j} + a_{i2}b_{2j}$$

Matrix Multiplication

- With matrices A, B

$$A = \begin{bmatrix} a_{00} & a_{10} & a_{20} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{00} & b_{10} & b_{20} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{bmatrix}$$

- To compute C=AB

$$c_{ij} = a_{i0}b_{0j} + a_{i1}b_{1j} + a_{i2}b_{2j}$$

- For example

$$c_{10} = a_{10}b_{00} + a_{11}b_{10} + a_{12}b_{20}$$