SOLUTIONS TO HOMEWORK 3
CMSC 441

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1. Homework 3 Solutions

Below are the solutions to the problems given in Homework 3. These problems can be found in the class handout [1].

2. Problem 2.3.7 on page 75 of Handout

Solve the following recursion using the charateristic equation method:

$$\begin{cases} T(n) = 2T(n/2) + \lg n, \text{ for } n \geq 2 \\ T(1) = 1 \end{cases}$$

Let $n = 2^m$. Hence, $m = \lg n$. Thus,

$$T\left(2^m\right) = 2T\left(2^{m-1}\right) + m$$

Let $S(M) = T\left(2^m\right)$. Thus,

$$S(m) = 2S(m-1) + m$$

Thus, we need to solve the inhomogeneous recursion

$$S(m) - 2(m-1) = p(m)b^m \ ,$$

where $p(m) = m$, and $b = 1$. So the charactersitic equation of this inhomogeneous system is

$$(x-2)(x-1)^2 = 0$$

with roots

$$1, 1, 2$$

Therefore,

$$S(m) = c_1 \cdot 1^m + c_2 \cdot m \cdot 1^m + c_3 \cdot 2^m = c_1 + c_2 m + c_3 2^m$$

We now find the constants c_1, c_2, c_3 using the following boundary conditions:

$$\begin{cases} S(0) = T\left(2^0\right) = T(1) = 1 \\ S(1) = 2S(1-1) + 1 = 3 \\ S(2) = 2S(2-1) + 2 = 8 \end{cases}$$

Hence, we need to solve the linear system

$$\begin{cases} c_1 + c_3 = 1 = S(0) \\ c_1 + c_2 + 2c_3 = 3 = S(1) \\ c_1 + 2c_2 + 4c_3 = 8 = S(2) \end{cases}$$

The solution to this linear sytem is

$$c_1 = -1,\ c_2 = -1,\ c_3 = 3$$

Thus,

$$T\left(2^m\right) = S(m) = -2 - m + 3 \cdot 2^m$$

Since $m = \lg n \iff n = 2^m$, we have

$$T(n) = -2 - \lg n + 3n = \Theta\left(n\right)$$

3. Problem 2.3.9 on page 76 of Handout

Solve the following recursion using the charateristic equation method:

$$\begin{cases} t_n = t_{n-1} + t_{n-3} - t_{n-4} \text{ , for } n \geq 4 \\ t_n = n \text{ , for } 0 \leq n \leq 3 \end{cases}$$

The characteristic equation of the homogeneous recursion

$$t_n - t_{n-1} - t_{n-3} + t_{n-4} = 0$$

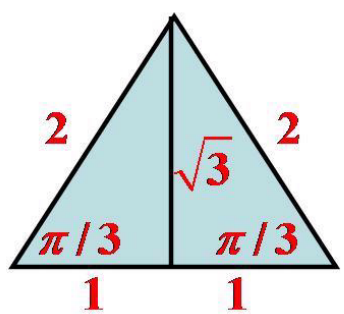
is

$$p(x) = x^4 - x^3 - x + 1 = (x-1)^2\left(x^2 + x + 1\right) = 0 \text{ .}$$

The roots of this system are

$$1,\ 1,\ \frac{-1 + i\sqrt{3}}{2},\ \frac{-1 - i\sqrt{3}}{2}$$

From the equilateral triangle



We know that $\cos(\pi/3) = \frac{1}{2}$and $\sin(\pi/3) = \frac{\sqrt{3}}{2}$. Hence, $e^{i\pi/3} = \frac{1+i\sqrt{3}}{2}$ and $e^{-i\pi/3} = \frac{1-i\sqrt{3}}{2}$. Thus,

$$\frac{-1 + i\sqrt{3}}{2} = -e^{-i\pi/3} = e^{i\pi} \cdot e^{-i\pi/3} = e^{2\pi i/3}$$

and

$$\frac{-1 - i\sqrt{3}}{2} = -e^{i\pi/3} = e^{i\pi} \cdot e^{i\pi/3} = e^{4\pi i/3} = e^{-2\pi i/3}$$

Thus, $\tau = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$ and $\tau^2 = e^{-2\pi i/3} = \frac{-1-i\sqrt{3}}{2}$ are 3rd roots of unity, and it can be easily verivied that $\tau^2 + \tau + 1 = 0$.

This the roots of the charateristic equation are

$$1,\ 1,\ \tau = e^{2\pi i/3},\ \tau^2 = \tau^{-1} = e^{-2\pi i/3}$$

Hence,

$$t_n = c_1 \cdot 1^n + c_2 \cdot n \cdot 1^n + c_3 \cdot \tau^n + c_4 \cdot \tau^{-n} = c_1 + c_2 n + c_3 \tau^n + c_4 \tau^{-n}$$

We now determine the constants c_1, c_2, c_3, c_4 with the boundary conditions by solving the following linear sytem of equations

$$\begin{cases} c_1 \quad\quad + c_3 \quad + c_4 \quad = 0 = t_0 \\ c_1 \quad + c_2 + c_3\tau \quad + c_4\tau^2 = 1 = t_1 \\ c_1 + 3c_2 + c_3\tau^2 + c_4\tau \quad = 2 = t_2 \\ c_1 + 3c_{c2} + c_3 \quad + c_4 \quad = 3 = t_3 \end{cases}$$

where we have used the fact that $\tau^3 = 1 = \tau^{-3}$. The solution to this system of linear equations is

$$c_1 = c_3 = c_4 = 0 \quad \text{and} \quad c_2 = 1\ .$$

Thus,

$$t_n = 0 + 1 \cdot n + 0 \cdot \tau^n + 0 \cdot \tau^{-n} = n = \Theta(n)\ .$$

4. Problem 2.3.10 on page 76 of Handout

Solve the following recursion using the charateristic equation method:

$$\begin{cases} T(n) = 5T(n/2) + (n \lg n)^2 \ \text{, for } n \geq 2 \\ T(1) = 1 \end{cases}$$

Let $n = 2^m$. Hence, $m = \lg n$. Thus,

$$T(2^m) = 5T\left(2^{m-1}\right) + (2^m \lg 2^m)^2$$

Let $S(M) = T(2^m)$. Thus,

$$S(m) = 5S(m-1) + m^2 4^m$$

Thus, we need to solve the inhomogeneous recursion

$$S(m) - 5(m-1) = p(m)b^m\ ,$$

where $p(m) = m^2$, and $b = 4$. So the charactersitic equation of this inhomogeneous system is

$$(x-5)(x-4)^3 = 0$$

with roots

$$4, 4, 4, 5$$

Hence,

$$S(m) = c_1 \cdot 4^m + c_2 \cdot m \cdot 4^m + c_3 \cdot m^2 \cdot 4^m + c_4 \cdot 5^m\ .$$

We now find the constants c_1, c_2, c_3, c_4 using the following boundary conditions:

$$\begin{cases} S(0) = T\left(2^0\right) = T(1) = 1 \\ S(1) = 5S(0) + 1^2 \cdot 4^1 = 5 + 5 = 9 \\ S(2) = 5S(1) + 2^2 \cdot 4^2 = 5 \cdot 9 + 64 = 109 \\ S(3) = 5S(2) + 3^2 \cdot 4^3 = 5 \cdot 109 + 9 \cdot 64 = 545 + 576 = 1121 \end{cases}$$

Hence, we need to solve the linear system

$$\begin{cases} c_1 + c_4 = 1 = S(0) \\ 4c_1 + 4c_2 + 4c_3 + 5c_4 = 9 = S(1) \\ 16c_1 + 32c_2 + 64c_3 + 25c_4 = 109 = S(2) \\ 64c_1 + 192c_2 + 576c_3 + 125c_4 = 1121 = S(3) \end{cases}$$

The solution to this linear sytem is

$$c_1 = -180,\ c_2 = -40,\ c_3 = -4,\ c_4 = 181$$

Thus, the solution to the recursion is

$$S(m) = -180 \cdot 4^m - 40 \cdot m \cdot 4^m - 4 \cdot m^2 \cdot 4^m + 181 \cdot 5^m = \Theta\left(5^m\right)$$

Since

$$5^m = 5^{\lg n} = 2^{(\lg 5)(\lg n)} = n^{\lg 5}\ ,$$

the solution to the original recursion is

$$T(n) = \Theta\left(n^{\lg 5}\right)$$

References

[1] Brassard, Gilles, and Paul Bratley, **"Algorithmics" Theory and Practice,"** Printice Hall, (1988), pages 65 -78.