ALGORITHMICS

Theory and Practice

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2.3 SOLVING RECURRENCES USING THE CHARACTERISTIC EQUATION

We have seen that the indispensable last step when analysing an algorithm is often to solve a system of recurrences. With a little experience and intuition such recurrences can often be solved by intelligent guesswork. This approach, which we do not illustrate here, generally proceeds in four stages: calculate the first few values of the recurrence, look for regularity, guess a suitable general form, and finally, prove by mathematical induction that this form is correct. Fortunately there exists a technique that can be used to solve certain classes of recurrence almost automatically.

2.3.1 Homogeneous Recurrences

Our starting point is the resolution of homogeneous linear recurrences with constant coefficients, that is, recurrences of the form

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = 0 \qquad (*)$$

where

i. the t_i are the values we are looking for. The recurrence is linear because it does not contain terms of the form $t_i\, t_{i+j}$, $t_i^{\,2}$, and so on;

ii. the coefficients a_i are constants; and

iii. the recurrence is homogeneous because the linear combination of the t_i is equal to zero.

After a while intuition may suggest we look for a solution of the form

$$t_n = x^n$$

where x is a constant as yet unknown. If we try this solution in (*), we obtain

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_k x^{n-k} = 0 .$$

This equation is satisfied if $x = 0$, a trivial solution of no interest, or else if

$$a_0 x^k + a_1 x^{k-1} + \cdots + a_k = 0 .$$

This equation of degree k in x is called the *characteristic equation* of the recurrence (*).

Suppose for the time being that the k roots $r_1, r_2, \ldots, r_k$ of this characteristic equation are all distinct (they could be complex numbers). It is then easy to verify that any linear combination

$$t_n = \sum_{i=1}^{k} c_i\, r_i^n$$

of terms r_i^n is a solution of the recurrence (*), where the k constants $c_1, c_2, \ldots, c_k$ are determined by the initial conditions. (We need exactly k initial conditions to determine the values of these k constants.) The remarkable fact, which we do not prove here, is that (*) has *only* solutions of this form.

Example 2.3.1. Consider the recurrence

$$t_n - 3t_{n-1} - 4t_{n-2} = 0 \qquad n \geq 2$$

subject to $t_0 = 0$, $t_1 = 1$.

The characteristic equation of the recurrence is

$$x^2 - 3x - 4 = 0$$

whose roots are -1 and 4. The general solution therefore has the form

$$t_n = c_1(-1)^n + c_2 4^n .$$

The initial conditions give

$$\begin{aligned} c_1 + c_2 &= 0 \qquad n = 0 \\ -c_1 + 4c_2 &= 1 \qquad n = 1 \end{aligned}$$

that is, $c_1 = -\frac{1}{5}$, $c_2 = \frac{1}{5}$.

We finally obtain

$$t_n = \tfrac{1}{5}[4^n - (-1)^n].$$

□

Example 2.3.2. Fibonacci. Consider the recurrence

$$t_n = t_{n-1} + t_{n-2} \qquad n \geq 2$$

subject to $t_0 = 0$, $t_1 = 1$.
(This is the definition of the Fibonacci sequence; see Section 1.7.5.)

The recurrence can be rewritten in the form $t_n - t_{n-1} - t_{n-2} = 0$, so the characteristic equation is

$$x^2 - x - 1 = 0$$

whose roots are

$$r_1 = \tfrac{1+\sqrt{5}}{2} \quad \text{and} \quad r_2 = \tfrac{1-\sqrt{5}}{2}.$$

The general solution is therefore of the form

$$t_n = c_1 r_1^n + c_2 r_2^n .$$

The initial conditions give

$$c_1 + \quad c_2 = 0 \qquad n = 0$$
$$r_1 c_1 + r_2 c_2 = 1 \qquad n = 1$$

from which it is easy to obtain

$$c_1 = \frac{1}{\sqrt{5}}, \qquad c_2 = -\frac{1}{\sqrt{5}}.$$

Thus $t_n = \frac{1}{\sqrt{5}}(r_1^n - r_2^n)$. To show that this is the same as the result obtained by De Moivre mentioned in Section 1.7.5, we need to note only that $r_1 = \phi$ and $r_2 = -\phi^{-1}$. □

* **Problem 2.3.1.** Consider the recurrence

$$t_n = 2t_{n-1} - 2t_{n-2} \qquad n \geq 2$$

subject to $t_0 = 0$, $t_1 = 1$.

Prove that $t_n = 2^{n/2}\sin(n\pi/4)$, not by mathematical induction but by using the characteristic equation. □

Now suppose that the roots of the characteristic equation are not all distinct. Let

$$p(x) = a_0 x^k + a_1 x^{k-1} + \cdots + a_k$$

be the polynomial in the characteristic equation, and let r be a multiple root. For every $n \geq k$, consider the nth degree polynomial defined by

$$h(x) = x[x^{n-k}p(x)]' = a_0 n x^n + a_1(n-1)x^{n-1} + \cdots + a_k(n-k)x^{n-k}.$$

Let $q(x)$ be the polynomial such that $p(x) = (x-r)^2 q(x)$. We have that

$$h(x) = x[(x-r)^2 x^{n-k} q(x)]' = x[2(x-r)x^{n-k}q(x) + (x-r)^2[x^{n-k}q(x)]'].$$

In particular, $h(r) = 0$. This shows that

$$a_0 n r^n + a_1(n-1)r^{n-1} + \cdots + a_k(n-k)r^{n-k} = 0,$$

that is, $t_n = nr^n$ is also a solution of (*). More generally, if m is the multiplicity of the root r, then $t_n = r^n$, $t_n = nr^n$, $t_n = n^2 r^n$, ..., $t_n = n^{m-1} r^n$ are all possible solutions of (*). The general solution is a linear combination of these terms and of the terms contributed by the other roots of the characteristic equation. Once again there are k constants to be determined by the initial conditions.

Example 2.3.3. Consider the recurrence

$$t_n = 5t_{n-1} - 8t_{n-2} + 4t_{n-3} \qquad n \geq 3$$

subject to $t_0 = 0$, $t_1 = 1$, $t_2 = 2$.

The recurrence can be written

$$t_n - 5t_{n-1} + 8t_{n-2} - 4t_{n-3} = 0$$

and so the characteristic equation is

$$x^3 - 5x^2 + 8x - 4 = 0$$

or $(x-1)(x-2)^2 = 0$.

The roots are 1 (of multiplicity 1) and 2 (of multiplicity 2). The general solution is therefore

$$t_n = c_1 1^n + c_2 2^n + c_3 n 2^n .$$

The initial conditions give

$$\begin{aligned} c_1 + c_2 \quad\quad &= 0 \qquad n = 0 \\ c_1 + 2c_2 + 2c_3 &= 1 \qquad n = 1 \\ c_1 + 4c_2 + 8c_3 &= 2 \qquad n = 2 \end{aligned}$$

from which we find $c_1 = -2$, $c_2 = 2$, $c_3 = -\frac{1}{2}$. Therefore

$$t_n = 2^{n+1} - n 2^{n-1} - 2 . \qquad \square$$

2.3.2 Inhomogeneous Recurrences

We now consider recurrences of a slightly more general form.

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = b^n p(n) \qquad (**)$$

The left-hand side is the same as (*), but on the right-hand side we have $b^n p(n)$, where

i. b is a constant; and
ii. $p(n)$ is a polynomial in n of degree d.

For example, the recurrence might be

$$t_n - 2t_{n-1} = 3^n .$$

In this case $b = 3$ and $p(n) = 1$, a polynomial of degree 0. A little manipulation allows us to reduce this example to the form (*). To see this, we first multiply the recurrence by 3, obtaining

$$3t_n - 6t_{n-1} = 3^{n+1} .$$

If we replace n by $n+1$ in the original recurrence, we get

$$t_{n+1} - 2t_n = 3^{n+1} .$$

Finally, subtracting these two equations, we have

$$t_{n+1} - 5t_n + 6t_{n-1} = 0,$$

which can be solved by the method of Section 2.3.1. The characteristic equation is

$$x^2 - 5x + 6 = 0$$

that is, $(x-2)(x-3) = 0$.

Intuitively we can see that the factor $(x-2)$ corresponds to the left-hand side of the original recurrence, whereas the factor $(x-3)$ has appeared as a result of our manipulation to get rid of the right-hand side.

Here is a second example.

$$t_n - 2t_{n-1} = (n+5)3^n$$

The necessary manipulation is a little more complicated: we must

a. multiply the recurrence by 9

b. replace n in the recurrence by $n+2$, and

c. replace n in the recurrence by $n+1$ and then multiply by -6,

obtaining respectively

$$\begin{aligned} 9t_n \quad - 18t_{n-1} &= \quad (n+5)3^{n+2} \\ t_{n+2} - \quad 2t_{n+1} &= \quad (n+7)3^{n+2} \\ -6t_{n+1} + 12t_n \quad &= -6(n+6)3^{n+1}. \end{aligned}$$

Adding these three equations, we obtain

$$t_{n+2} - 8t_{n+1} + 21t_n - 18t_{n-1} = 0.$$

The characteristic equation of this new recurrence is

$$x^3 - 8x^2 + 21x - 18 = 0$$

that is, $(x-2)(x-3)^2 = 0$.

Once again, we can see that the factor $(x-2)$ comes from the left-hand side of the original recurrence, whereas the factor $(x-3)^2$ is the result of our manipulation.

Generalizing this approach, we can show that to solve (**) it is sufficient to take the following characteristic equation:

$$(a_0x^k + a_1x^{k-1} + \cdots + a_k)(x-b)^{d+1} = 0.$$

Once this equation is obtained, proceed as in the homogeneous case.

Example 2.3.4. The number of movements of a ring required in the Towers of Hanoi problem (see Example 2.2.11) is given by

$$t_n = 2t_{n-1} + 1 \qquad n \ge 1$$

subject to $t_0 = 0$.

The recurrence can be written

$$t_n - 2t_{n-1} = 1,$$

which is of the form (**) with $b = 1$ and $p(n) = 1$, a polynomial of degree 0. The characteristic equation is therefore

$$(x-2)(x-1) = 0$$

where the factor $(x-2)$ comes from the left-hand side and the factor $(x-1)$ comes from the right-hand side. The roots of this equation are 1 and 2, so the general solution of the recurrence is

$$t_n = c_1 1^n + c_2 2^n .$$

We need two initial conditions. We know that $t_0 = 0$; to find a second initial condition we use the recurrence itself to calculate

$$t_1 = 2t_0 + 1 = 1.$$

We finally have

$$\begin{aligned} c_1 + \ c_2 &= 0 \qquad n = 0 \\ c_1 + 2c_2 &= 1 \qquad n = 1 \end{aligned}$$

from which we obtain the solution

$$t_n = 2^n - 1.$$ □

If all we want is the order of t_n, there is no need to calculate the constants in the general solution. In the previous example, once we know that

$$t_n = c_1 1^n + c_2 2^n$$

we can already conclude that $t_n \in \Theta(2^n)$. For this it is sufficient to notice that t_n, the number of movements of a ring required, is certainly neither negative nor a constant, since clearly $t_n \ge n$. Therefore $c_2 > 0$, and the conclusion follows.

In fact we can obtain a little more. Substituting the general solution back into the original recurrence, we find

$$\begin{aligned} 1 &= t_n - 2t_{n-1} \\ &= c_1 + c_2 2^n - 2(c_1 + c_2 2^{n-1}) \\ &= -c_1 . \end{aligned}$$

Whatever the initial condition, it is therefore always the case that c_1 must be equal to -1.

Problem 2.3.2. There is nothing surprising in the fact that we can determine one of the constants in the general solution without looking at the initial condition; on the contrary! Why? □

Example 2.3.5. Consider the recurrence

$$t_n = 2t_{n-1} + n.$$

This can be written

$$t_n - 2t_{n-1} = n,$$

which is of the form (**) with $b = 1$ and $p(n) = n$, a polynomial of degree 1. The characteristic equation is therefore

$$(x-2)(x-1)^2 = 0$$

with roots 2 (multiplicity 1) and 1 (multiplicity 2). The general solution is

$$t_n = c_1 2^n + c_2 1^n + c_3 n\, 1^n.$$

In the problems that interest us, we are always looking for a solution where $t_n \geq 0$ for every n. If this is so, we can conclude immediately that t_n must be in $O(2^n)$. □

Problem 2.3.3. By substituting the general solution back into the recurrence, prove that in the preceding example $c_2 = -2$ and $c_3 = -1$ whatever the initial condition. Conclude that all the interesting solutions of the recurrence must have $c_1 > 0$, and hence that they are all in $\Theta(2^n)$. □

A further generalization of the same type of argument allows us finally to solve recurrences of the form

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = b_1^n p_1(n) + b_2^n p_2(n) + \cdots \qquad (***)$$

where the b_i are distinct constants and the $p_i(n)$ are polynomials in n respectively of degree d_i. It suffices to write the characteristic equation

$$(a_0 x^k + a_1 x^{k-1} + \cdots + a_k)(x-b_1)^{d_1+1}(x-b_2)^{d_2+1} \cdots = 0,$$

which contains one factor corresponding to the left-hand side and one factor corresponding to each term on the right-hand side, and to solve the problem as before.

Example 2.3.6. Solve

$$t_n = 2t_{n-1} + n + 2^n \qquad n \geq 1$$

subject to $t_0 = 0$.

The recurrence can be written

$$t_n - 2t_{n-1} = n + 2^n,$$

which is of the form (***) with $b_1 = 1$, $p_1(n) = n$, $b_2 = 2$, $p_2(n) = 1$. The degree of $p_1(n)$ is 1, and $p_2(n)$ is of degree 0. The characteristic equation is

$$(x-2)(x-1)^2(x-2) = 0,$$

which has roots 1 and 2, both of multiplicity 2. The general solution of the recurrence is therefore of the form

$$t_n = c_1 1^n + c_2 n 1^n + c_3 2^n + c_4 n 2^n .$$

Using the recurrence, we can calculate $t_1 = 3$, $t_2 = 12$, $t_3 = 35$. We can now determine c_1, c_2, c_3 and c_4 from

$$\begin{array}{llll} c_1 & + c_3 & = 0 & n = 0 \\ c_1 + c_2 + 2c_3 + & 2c_4 & = 3 & n = 1 \\ c_1 + 2c_2 + 4c_3 + & 8c_4 & = 12 & n = 2 \\ c_1 + 3c_2 + 8c_3 + & 24c_4 & = 35 & n = 3 \end{array}$$

arriving finally at

$$t_n = -2 - n + 2^{n+1} + n 2^n .$$

We could obviously have concluded that $t_n \in O(n 2^n)$ without calculating the constants. □

Problem 2.3.4. Prove that all the solutions of this recurrence are in fact in $\Theta(n 2^n)$, regardless of the initial condition. □

Problem 2.3.5. If the characteristic equation of the recurrence (***) is of degree

$$m = k + (d_1+1) + (d_2+1) + \cdots ,$$

then the general solution contains m constants $c_1, c_2, \ldots, c_m$. How many constraints on these constants can be obtained without using the initial conditions? (See Problems 2.3.3 and 2.3.4.) □

2.3.3 Change of Variable

It is sometimes possible to solve more complicated recurrences by making a change of variable. In the following examples we write $T(n)$ for the term of a general recurrence, and t_k for the term of a new recurrence obtained by a change of variable.

Example 2.3.7. Here is how we can find the order of $T(n)$ if n is a power of 2 and if

$$T(n) = 4T(n/2) + n \qquad n > 1.$$

Replace n by 2^k (so that $k = \lg n$) to obtain $T(2^k) = 4T(2^{k-1}) + 2^k$. This can be written

$$t_k = 4t_{k-1} + 2^k$$

if $t_k = T(2^k) = T(n)$. We know how to solve this new recurrence: the characteristic equation is

$$(x-4)(x-2) = 0$$

and hence $t_k = c_1 4^k + c_2 2^k$.

Putting n back instead of k, we find

$$T(n) = c_1 n^2 + c_2 n.$$

$T(n)$ is therefore in $O(n^2 \mid n$ is a power of 2). □

Example 2.3.8. Here is how to find the order of $T(n)$ if n is a power of 2 and if

$$T(n) = 4T(n/2) + n^2 \qquad n > 1.$$

Proceeding in the same way, we obtain successively

$$T(2^k) = 4T(2^{k-1}) + 4^k$$

$$t_k = 4t_{k-1} + 4^k.$$

The characteristic equation is $(x-4)^2 = 0$, and so

$$t_k = c_1 4^k + c_2 k 4^k$$

$$T(n) = c_1 n^2 + c_2 n^2 \lg n.$$

Thus $T(n) \in O(n^2 \log n \mid n$ is a power of 2). □

Example 2.3.9. Here is how to find the order of $T(n)$ if n is a power of 2 and if

$$T(n) = 2T(n/2) + n \lg n \qquad n > 1.$$

As before, we obtain

$$T(2^k) = 2T(2^{k-1}) + k\,2^k$$

$$t_k = 2t_{k-1} + k\,2^k.$$

The characteristic equation is $(x-2)^3 = 0$, and so

$$t_k = c_1 2^k + c_2 k\,2^k + c_3 k^2 2^k$$

$$T(n) = c_1 n + c_2 n \lg n + c_3 n \lg^2 n.$$

Hence, $T(n) \in O(n \log^2 n \mid n$ is a power of 2). □

Example 2.3.10. We want to find the order of $T(n)$ if n is a power of 2 and if

$$T(n) = 3T(n/2) + cn \qquad (c \text{ is constant, } n = 2^k > 1).$$

We obtain successively

$$T(2^k) = 3T(2^{k-1}) + c\,2^k$$

$$t_k = 3t_{k-1} + c\,2^k.$$

The characteristic equation is $(x-3)(x-2) = 0$, and so

$$t_k = c_1 3^k + c_2 2^k$$

$$T(n) = c_1 3^{\lg n} + c_2 n$$

and hence since $a^{\lg b} = b^{\lg a}$

$$T(n) = c_1 n^{\lg 3} + c_2 n.$$

Finally, $T(n) \in O(n^{\lg 3} \mid n \text{ is a power of 2})$. □

Remark. In Examples 2.3.7 to 2.3.10 the recurrence given for $T(n)$ only applies when n is a power of 2. It is therefore inevitable that the solution obtained should be in conditional asymptotic notation. In each of these four cases, however, it is sufficient to add the condition that $T(n)$ is eventually nondecreasing to be able to conclude that the asymptotic results obtained apply unconditionally for all values of n. This follows from problem 2.1.20 since the functions n^2, $n^2 \log n$, $n \log^2 n$ and $n^{\lg 3}$ are smooth.

* **Problem 2.3.6.** The constants $n_0 \geq 1$, $b \geq 2$ and $k \geq 0$ are integers, whereas a and c are positive real numbers. Let $T : \mathbb{N} \to \mathbb{R}^+$ be an eventually nondecreasing function such that

$$T(n) = aT(n/b) + cn^k \qquad n > n_0$$

when n/n_0 is a power of b. Show that the exact order of $T(n)$ is given by

$$T(n) \in \begin{cases} \Theta(n^k) & \text{if } a < b^k \\ \Theta(n^k \log n) & \text{if } a = b^k \\ \Theta(n^{\log_b a}) & \text{if } a > b^k. \end{cases}$$

Rather than proving this result by constructive induction, obtain it using the techniques of the characteristic equation and change of variable. This result is generalized in Problem 2.3.13. □

Problem 2.3.7. Solve the following recurrence exactly for n a power of 2:

$$T(n) = 2T(n/2) + \lg n \qquad n \geq 2$$

subject to $T(1) = 1$.

Express your solution as simply as possible using the Θ notation. □

Problem 2.3.8. Solve the following recurrence exactly for n of the form 2^{2^k}:

$$T(n) = 2T(\sqrt{n}\,) + \lg n \qquad n \geq 4$$

subject to $T(2) = 1$.

Express your solution as simply as possible using the Θ notation. □

2.3.4 Range Transformations

When we make a change of variable, we transform the domain of the recurrence. It is sometimes useful to transform the range instead in order to obtain something of the form (***). We give just one example of this approach. We want to solve

$$T(n) = n\,T^2(n/2) \qquad n > 1$$

subject to $T(1) = 6$ for the case when n is a power of 2. The first step is a change of variable: put $t_k = T(2^k)$, which gives

$$t_k = 2^k\, t_{k-1}^2 \qquad k > 0$$

subject to $t_0 = 6$.

At first glance, none of the techniques we have seen applies to this recurrence since it is not linear, and furthermore, one of the coefficients is not constant. To transform the range, we create a new recurrence by putting $V_k = \lg t_k$, which yields

$$V_k = k + 2V_{k-1} \qquad k > 0$$

subject to $V_0 = \lg 6$.

The characteristic equation is $(x-2)(x-1)^2 = 0$, and so

$$V_k = c_1 2^k + c_2 1^k + c_3 k\, 1^k.$$

From $V_0 = 1 + \lg 3$, $V_1 = 3 + 2\lg 3$, and $V_2 = 8 + 4\lg 3$ we obtain $c_1 = 3 + \lg 3$, $c_2 = -2$, and $c_3 = -1$, and hence

$$V_k = (3 + \lg 3)2^k - k - 2.$$

Finally, using $t_k = 2^{V_k}$ and $T(n) = t_{\lg n}$, we obtain

$$T(n) = \frac{2^{3n-2}3^n}{n}.$$

2.3.5 Supplementary Problems

Problem 2.3.9. Solve the following recurrence exactly:

$$t_n = t_{n-1} + t_{n-3} - t_{n-4} \qquad n \geq 4$$

subject to $t_n = n$ for $0 \leq n \leq 3$. Express your answer as simply as possible using the Θ notation. □

Problem 2.3.10. Solve the following recurrence exactly for n a power of 2:

$$T(n) = 5T(n/2) + (n \lg n)^2 \qquad n \geq 2$$

subject to $T(1) = 1$. Express your answer as simply as possible using the Θ notation. □

Problem 2.3.11. (Multiplication of large integers: see Sections 1.7.2, 4.1, and 4.7) Consider any constants $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$. Let $T : \mathbb{N} \to \mathbb{R}^*$ be an eventually nondecreasing function such that

$$T(n) \leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(1 + \lceil n/2 \rceil) + cn \qquad n > n_0 .$$

Prove that $T(n) \in O(n^{\lg 3})$. *Hint*: observe that $T(n) \leq 3T(1 + \lceil n/2 \rceil) + cn$ for $n > n_0$, make the change of variable $T'(n) = T(n+2)$, use Example 2.3.10 to solve for $T'(n)$ when n is a power of 2, and use problem 2.1.20 to conclude for $T(n)$. □

Problem 2.3.12. Solve the following recurrence exactly:

$$t_n = t_{n-1} + 2t_{n-2} - 2t_{n-3} \qquad n \geq 3$$

subject to $t_n = 9n^2 - 15n + 106$ for $0 \leq n \leq 2$. Express your answer as simply as possible using the Θ notation. □

*** Problem 2.3.13.** Recurrences arising from the analysis of divide-and-conquer algorithms (Chapter 4) can usually be handled by Problem 2.3.6. In some cases, however, a more general result is required (and the technique of the characteristic equation does not always apply).

The constants $n_0 \geq 1$ and $b \geq 2$ are integers, whereas a and d are real positive constants. Define

$$X = \{ n \in \mathbb{N} \mid \log_b(n/n_0) \in \mathbb{N} \} = \{ n \in \mathbb{N} \mid (\exists i \in \mathbb{N}) [n = n_0 b^i] \} .$$

Let $f : X \to \mathbb{R}^*$ be an arbitrary function. Define the function $T : X \to \mathbb{R}^*$ by the recurrence

$$T(n) = \begin{cases} d & \text{if } n = n_0 \\ a\,T(n/b) + f(n) & \text{if } n \in X,\ n > n_0 . \end{cases}$$

Let $p = \log_b a$. It turns out that the simplest way to express $T(n)$ in asymptotic notation depends on how $f(n)$ compares to n^p. In what follows, all asymptotic notation is implicitly conditional on $n \in X$. Prove that

i. If we set $f(n_0) = d$ (which is of no consequence for the definition of T), the value of $T(n)$ is given by a simple summation when $n \in X$:

$$T(n) = \sum_{i=0}^{\log_b(n/n_0)} a^i f(n/b^i) \ .$$

ii. Let q be any strictly positive real constant; then

$$T(n) \in \begin{cases} \Theta(n^p) & \text{if } f(n) \in O(n^p/(\log n)^{1+q}) \\ \Theta(f(n)\log n \log\log n) & \text{if } f(n) \in \Theta(n^p/\log n) \\ \Theta(f(n)\log n) & \text{if } f(n) \in \Theta(n^p(\log n)^{q-1}) \\ \Theta(f(n)) & \text{if } f(n) \in \Theta(n^{p+q}) \ . \end{cases}$$

Note that the third alternative includes $f(n) \in \Theta(n^p)$ by choosing $q = 1$.

iii. As a special case of the first alternative, $T(n) \in \Theta(n^p)$ whenever $f(n) \in O(n^r)$ for some real constant $r < p$.

iv. The last alternative can be generalized to include cases such as $f(n) \in \Theta(n^{p+q}\log n)$ or $f(n) \in \Theta(n^{p+q}/\log n)$; we also get $T(n) \in \Theta(f(n))$ if there exist a function $g: X \to \mathbb{R}^*$ and a real constant $\alpha > a$ such that $f(n) \in \Theta(g(n))$ and $g(bn) \geq \alpha g(n)$ for all $n \in X$.

** **v.** Prove or disprove that the third alternative can be generalized as follows: $T(n) \in \Theta(f(n)\log n)$ whenever there exist two strictly positive real constants $q_1 \leq q_2$ such that $f(n) \in O(n^p(\log n)^{q_2-1})$ and $f(n) \in \Omega(n^p(\log n)^{q_1-1})$. If you disprove it, find the simplest but most general additional constraint on $f(n)$ that suffices to imply $T(n) \in \Theta(f(n)\log n)$. □

Problem 2.3.14. Solve the following recurrence exactly:

$$t_n = 1/(4 - t_{n-1}) \qquad n > 1$$

subject to $t_1 = 1/4$. □

Problem 2.3.15. Solve the following recurrence exactly as a function of the initial conditions a and b:

$$T(n+2) = (1 + T(n+1))/T(n) \qquad n \geq 2$$

subject to $T(0) = a$, $T(1) = b$. □

Problem 2.3.16. Solve the following recurrence exactly:

$$T(n) = \tfrac{3}{2}T(n/2) - \tfrac{1}{2}T(n/4) - 1/n \qquad n \geq 3$$

subject to $T(1) = 1$ and $T(2) = 3/2$. □

2.4 REFERENCES AND FURTHER READING

The asymptotic notation has existed for some while in mathematics: see Bachmann (1894) and de Bruijn (1961). Knuth (1976) gives an account of its history and proposes a standard form for it. Conditional asymptotic notation and its use in Problem 2.1.20 are introduced by Brassard (1985), who also suggests that "one-way inequalities" should be abandoned in favour of a notation based on sets. For information on calculating limits and on de l'Hôpital's rule, consult any book on mathematical analysis, Rudin (1953), for instance.

The book by Purdom and Brown (1985) presents a number of techniques for analysing algorithms. The main mathematical aspects of the analysis of algorithms can also be found in Greene and Knuth (1981).

Example 2.1.1 corresponds to the algorithm of Dixon (1981). Problem 2.2.3 comes from Williams (1964). The analysis of disjoint set structures given in Example 2.2.10 is adapted from Hopcroft and Ullman (1973). The more precise analysis making use of Ackermann's function can be found in Tarjan (1975, 1983). Buneman and Levy (1980) and Dewdney (1984) give a solution to Problem 2.2.15.

Several techniques for solving recurrences, including the characteristic equation and change of variable, are explained in Lueker (1980). For a more rigorous mathematical treatment see Knuth (1968) or Purdom and Brown (1985). The paper by Bentley, Haken, and Saxe (1980) is particularly relevant for recurrences occurring from the analysis of divide-and-conquer algorithms (see Chapter 4).