STUDY PROBLEMS FOR EXAM I
CMSC 203
DISCRETE STRUCTURES

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1. Use the principle of mathematical induction to prove that

$$P(n): \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6},$$

for all integers $n \geq 1$.

Answer:

Proof (by weak induction):

Basis Step: $P(n)$ is true for $n = 1$, for:

$$\sum_{j=1}^{1} j^2 = 1^2 = 1 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$$

Inductive Hypothesis: Assume for a fixed but arbitrary integer $k \geq 1$ that $P(k)$ is true, i.e., that

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

Inductive Step: We wish to use the Inductive Hypothesis to show that $P(k+1)$ is true, i.e., that

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

[We start with the left hand side and transform it using the inductive hypothesis into the right hand side.]

$$\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= \left(\sum_{j=1}^{k} j^2\right) + (k+1)^2 && \textbf{Reason: } \text{Basic algebra} \\
&= \tfrac{k(k+1)(2k+1)}{6} + (k+1)^2 && \textbf{Reason: } \text{Ind. Hypoth. \&substitution} \\
&= \tfrac{k(k+1)(2k+1)+6(k+1)^2}{6} && \textbf{Reason: } \text{Basic algebra} \\
&= \tfrac{(k+1)}{6}\left[k(2k+1) + 6(k+1)\right] && \textbf{Reason: } \text{Basic algebra} \\
&= \tfrac{(k+1)}{6}\left[2k^2 + k + 6k + 6\right] && \textbf{Reason: } \text{Basic algebra} \\
&= \tfrac{(k+1)}{6}\left(2k^2 + 7k + 6\right) && \textbf{Reason: } \text{Basic algebra} \\
&= \tfrac{(k+1)}{6}(k+2)(2k+3) && \textbf{Reason: } \text{Basic algebra} \\
&= \tfrac{(k+1)[(k+1)+1][2(k+1)+1]}{6} && \textbf{Reason: } \text{Basic algebra}
\end{aligned}$$

Thus, we have used the inductive hypothesis to prove that

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)\left[(k+1)+1\right]\left[2(k+1)+1\right]}{6}$$

Magic Wand Step: By the P.M.I.,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \geq 1$$

$\square$

Q.E.D.

2. Use the principle of mathematical induction to prove that

$$\prod_{j=2}^{n}\left(1 - \frac{1}{j^2}\right) = \frac{n+1}{2n}$$

for all integers $n \geq 2$.

3. Let d_1, d_2, d_3, ... be the sequence defined by

$$d_j = d_{j-1} \cdot d_{j-2} \text{ for all integers } j \geq 3$$

and

$$d_1 = \frac{9}{10} \quad \text{and} \quad d_2 = \frac{10}{11}$$

Use math induction to prove that

$$P(n) : d_n \leq 1 \text{ for all integers } n \geq 1.$$

Proof (by strong induction):

Basis Step: Both $P(1)$ and $P(2)$ are true, for:

$$\begin{cases} d_1 = \frac{9}{10} \leq 1 & \textbf{Reason: } \text{Definition of } d_1 \\ \\ d_2 = \frac{10}{11} \leq 1 & \textbf{Reason: } \text{Definition of } d_2 \end{cases}$$

Inductive Hypothesis: Assume for a fixed but arbitrary integer $k > 2$ that $P(\ell)$ is true for $1 \leq \ell < k$, i.e., that

$$d_\ell \leq 1 \text{ for } 1 \leq \ell < k$$

Inductive Step: We wish to use the Inductive Hypothesis to show that $P(k+1)$ is true, i.e., that

$$d_k \leq 1$$

$$\begin{array}{lll} & d_k = d_{k-1} \cdot d_{k-2} & \textbf{Reason: } \text{Definition of } d_k \\ \\ \text{But} & d_{k-1} \leq 1 \text{ and } d_{k-2} \leq 1 & \textbf{Reason: } \text{Ind. Hypoth.} \\ \\ \text{thus,} & d_k \leq 1 & \textbf{Reason: } \text{Basic algebra} \end{array}$$

Magic Wand Step: Hence, by. the P.M.I.,

$$d_n \leq 1 \quad \text{for} \quad n \geq 1$$

□

Q.E.D.

4. Let e_0, e_1, e_2, ... be the sequence defined by

$$e_j = e_{j-1} + e_{j-2} + e_{j-3} \quad \text{for all integers } j \geq 3$$

and

$$e_0 = 1, \quad e_1 = 2, \quad e_2 = 3$$

Use math induction to prove that $e_n \leq 3^n$ for all integers $n \geq 0$.

5. What is the definition of universal modus ponens? See page 112 of text.

6. What is the definition of universal modus tollens? See page 114 of text.

7. Give the contrapositive, converse, and inverse of the following statement

$$\forall x \in \mathbb{R}, \ [x\,(x+1) > 0] \longrightarrow [(x > 0) \vee (x < -1)]$$

Also, give the negation of the above statement.

Contrapositive:

$$\forall x \in \mathbb{R}, \ [(x \leq 0) \wedge (x \geq -1)] \longrightarrow [x\,(x+1) \leq 0]$$

Converse:

$$\forall x \in \mathbb{R}, [(x > 0) \vee (x < -1)] \longrightarrow [x\,(x+1) > 0]$$

Inverse:

$$\forall x \in \mathbb{R}, \ [x\,(x+1) \leq 0] \longrightarrow [(x \leq 0) \wedge (x \geq -1)]$$

Negation

$$\exists x \in \mathbb{R}, \ [x\,(x+1) > 0] \wedge [(x \leq 0) \wedge (x \geq -1)]$$

Remark: Please note that $p \longrightarrow q \equiv \sim p \vee q$.. Hence, $\sim (p \longrightarrow q) \equiv p \wedge \sim q$

8. In the following statement, push the negation "$\sim$" to the right of all quantifiers

$$\sim \forall x \forall y, P(x,y)$$

The answer is

$$\exists x \exists y, \sim P(x,y)$$

because

$$\sim \forall x \forall y, P(x,y) \equiv \exists x \sim \forall y, P(x,y) \equiv \exists x \exists y, \sim P(x,y)$$

9. The following "proof" that every integer is rational is incorrect. Find the mistake.

"Proof" (by contradiction):Suppose not. Suppose every integer is irrational. Then the integer 1 *is irrational. But* $1 = \frac{1}{1}$*, which is rational. This is a contradiction. Hence, the supposition is false, and the theorem is true. Q.E.D."*

10. Use a truth table to verify that

$$p \longrightarrow (q \longrightarrow r) \equiv (p \wedge q) \longrightarrow r$$

are logically equivalent. The answer is:

p	q	r	$q \longrightarrow r$	$p \wedge q$	$p \longrightarrow (q \longrightarrow r)$	$(p \wedge q) \longrightarrow r$
T	T	T	T	T	T	T
T	T	F	F	T	F	F
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	F	T	T
F	T	F	F	F	T	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

Since the last two columns are identical, the statements are logically equivalent.

11. Use a truth table to determine whether or not the following argument is valid. Be sure to state why your truth table justifies your answer.

$$\begin{array}{ll} & p \wedge q \longrightarrow \sim r \\ & p \vee \sim q \\ & \sim q \longrightarrow p \\ \therefore & \sim r \end{array}$$

The answer is:

p	q	r	$\sim q$	$\sim r$	$p \wedge q$	$p \wedge q \longrightarrow \sim r$	$p \vee \sim q$	$\sim q \longrightarrow p$	$\sim r$	
T	T	T	F	F	T	F	T	T	F	
T	T	F	F	T	T	T	T	T	T	OK
T	F	T	T	F	F	T	T	T	F	NOT OK
T	F	F	T	T	F	T	T	T	T	OK
F	T	T	F	F	F	T	F	T	F	
F	T	F	F	T	F	T	F	T	T	
F	F	T	T	F	F	T	T	F	F	
F	F	F	T	T	F	T	T	F	T	

For this to be a valid argument, the table must show that, whenever all premises are true, the conclusion is true. But the table indicates this is not true when $pqr = TFT$. Hence, this is an invalid argument.

12. Use a truth table to determine whether or not the following are logically equivalent:

$$\sim (p \longrightarrow q) \quad \text{and} \quad p\wedge \ \sim q$$

Be sure to state why your truth table justifies your answer.

13. Use the rules of inference to show that the following is a valid argument. Do not use a truth table.

$$\begin{array}{rl} & p \longrightarrow q \\ & \sim q \\ & s \longrightarrow p \\ & r \vee s \\ & r \longrightarrow t \\ \therefore & t \end{array}$$

The answer is

$$\left.\begin{array}{rl} & p \longrightarrow q \\ & \sim q \\ \therefore & \sim p \end{array}\right\} \text{Modus Tollens}$$

$$\left.\begin{array}{rl} & s \longrightarrow p \\ & \sim p \\ \therefore & \sim s \end{array}\right\} \text{Modus Tollens}$$

$$\left.\begin{array}{rl} & r \vee s \\ & \sim s \\ \therefore & r \end{array}\right\} \text{Ellimination}$$

$$\left.\begin{array}{rl} & r \longrightarrow t \\ & r \\ \therefore & t \end{array}\right\} \text{Modus Ponens}$$

14. Write the base 10 number 23 as a binary number.

15. Given that $\sqrt{2}$ is irrational, use proof-by-contradiction to prove that $3+5\sqrt{2}$ is also irrational. The answer can be found on page 181 of the text.

16. Prove that the difference of any two rational numbers is a rational number. The proof is similar to the proof of Theorem 3.2.2 found on page 145 of the text.

17. Prove in two ways, i.e., by contradiction and by contraposition, that

$$\forall n \in \mathbb{Z}, \text{ if } n^2 \text{ is odd, then } n \text{ is odd.}$$

The proof by contradiction is very much like the proof of Proposition 3.6.4 found on page 177 of the text. The proof by contraposion is simply proving that the contraposive

$$\forall n \in \mathbb{Z}, \text{ if } n \text{ is even, then } n^2 \text{ is even.}$$

is true.

18. Simplify the followng product as much as possible

$$\prod_{j=1}^{n} \left(2 \cdot 4^j\right)$$

The answer is

$$\prod_{j=1}^{n} \left(2 \cdot 4^j\right) = \left(\prod_{j=1}^{n} 2\right) \cdot \left(\prod_{j=1}^{n} 4^j\right) = 2^n \cdot 4^{\sum_{j=1}^{n} j}$$

But $\sum_{j=1}^{n} j$ is the arithmetic series. Hence, $\sum_{j=1}^{n} j = n\left(n+1\right)/2$. Thus,

$$\prod_{j=1}^{n} \left(2 \cdot 4^j\right) = 2^n \cdot 4^{\sum_{j=1}^{n} j} = 2^n \cdot 4^{n(n+1)/2} = 2^n \cdot 2^{2[n(n+1)/2]} = 2^n \cdot 2^{n(n+1)} = 2^{n^2+2n}$$

19. Simplify the following sum as much as possible

$$\sum_{j=0}^{n} (2+4k)$$

The anser is

$$\sum_{j=0}^{n} (2+4k) = \sum_{j=0}^{n} 2 + \sum_{j=0}^{n} 4k = 2(n+1) + 4\sum_{j=0}^{n} k = 4 \cdot \frac{n\left(n+1\right)}{2} = 2n\left(n+1\right)$$

20. Transform the following product by making the change of variable $i = k+1$

$$\prod_{k=1}^{n} \frac{k}{k^2+4}$$

The answer is:

$$\prod_{k=1}^{n} \frac{k}{k^2+4} = \prod_{i=2}^{n+1} \frac{i-1}{(i-1)^2+4} = \prod_{i=2}^{n+1} \frac{i-1}{i^2-2i+5}$$

21. Simplify the following product as much as possible

$$\prod_{j=2}^{5} \frac{(j-1)\cdot j}{(j+1)\cdot(j+2)}$$

The answer is:

$$\prod_{j=2}^{5} \frac{(j-1)\cdot j}{(j+1)\cdot(j+2)} = \left(\frac{1\cdot 2}{3\cdot 4}\right)\cdot\left(\frac{2\cdot 3}{4\cdot 5}\right)\cdot\left(\frac{3\cdot 4}{5\cdot 6}\right)\cdot\left(\frac{4\cdot 5}{6\cdot 7}\right)$$

which simplifies to

$$\frac{1}{3\cdot 5\cdot 7} = \frac{1}{105}$$

22. Determine whether or not the following argument is valid. Then state the reason for your answer.

$$\begin{array}{ll} & \text{All honest people pay their taxes.} \\ & \text{Darth is not honest.} \\ \therefore & \text{Darth does not pay his taxes.} \end{array}$$

For answer, see page 118 of text.