CMSC 643
RESEARCH PROBLEM 1
QUANTUM TOMOGRAPHY FOR A SINGLE QUBIT

DR. LOMONACO

1. Setting up the problem

Problem 1. *Given many copies of a qubit in an unknown fixed state*

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\ ,$$

where

$$|a|^2 + |b|^2 = 1\ ,$$

find a way to estimate the amplitudes a *and* b *to arbitrary accuracy.*

Since an overall phase can not be physically determined, we can without loss of generality assume that the amplitude a is a non-negative real number.

We begin by showing how to estimate the amplitudes a and b to arbitray accuracy by measuring many copies of the state $|\psi\rangle$ with respect to the Pauli spin operators σ_1, σ_2, σ_3. These observables together with their respective eigenvalues and corresponding eigenkets are given in the table below:

Eigenvalue	$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$+1$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\|0\rangle + \|1\rangle}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\|0\rangle + i\|1\rangle}{\sqrt{2}}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \|0\rangle$
-1	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\|0\rangle - \|1\rangle}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{\|0\rangle - i\|1\rangle}{\sqrt{2}}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \|1\rangle$

The algebraic expressions for the ket $|\psi\rangle$, expressed in terms of the eigenstates of each of the operators σ_1, σ_2, σ_3, are given respectively by

$$\begin{cases} |\psi\rangle = \left(\frac{a+b}{\sqrt{2}}\right)\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) + \left(\frac{a-b}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) \\ \\ |\psi\rangle = \left(\frac{a-ib}{\sqrt{2}}\right)\left(\frac{|0\rangle+i|1\rangle}{\sqrt{2}}\right) + \left(\frac{a+ib}{\sqrt{2}}\right)\left(\frac{|0\rangle-i|1\rangle}{\sqrt{2}}\right) \\ \\ |\psi\rangle = a\,|0\rangle + b\,|1\rangle \end{cases}$$

Thus, the probabilities p_1^+, p_2^+, p_3^+ of producing the eigenvalue $+1$ when measuring $|\psi\rangle$ with respect to each of the observables σ_1, σ_2, σ_3 are given respectively by:

$$\begin{cases} p_1^+ = \left|\frac{a+b}{\sqrt{2}}\right|^2 = \frac{1}{2} + \frac{\overline{a}b + a\overline{b}}{2} = \frac{1}{2} + a\left(\frac{b+\overline{b}}{2}\right) = \frac{1}{2} + a\,\mathrm{Re}\,(b) \\ \\ p_2^+ = \left|\frac{a-ib}{\sqrt{2}}\right|^2 = \frac{1}{2} + \frac{\overline{a}b - a\overline{b}}{2i} = \frac{1}{2} + a\left(\frac{b-\overline{b}}{2i}\right) = \frac{1}{2} + a\,\mathrm{Im}\,(b) \\ \\ p_3^+ = |a|^2 \end{cases}$$

It now follows that if the probabilities p_1^+, p_2^+, p_3^+ are known, then the amplitudes a and b can be computed from the following expressions:

$$\begin{cases} a \quad = \sqrt{p_3^+} \\ \\ \mathrm{Re}\,(b) = \frac{\left(p_1^+ - \frac{1}{2}\right)}{\sqrt{p_3^+}} \\ \\ \mathrm{Im}\,(b) = \frac{\left(p_2^+ - \frac{1}{2}\right)}{\sqrt{p_3^+}} \end{cases}$$

Your *"Mission Impossible Assignment,"* should you choose to accept, is to accomplish the following two tasks.

TASK 1:

Write a Maple subroutine

$$\textsc{Meas}(\psi, \mathcal{O})$$

that takes as input an arbitrary **unknown** 2-qubit ket $|\psi\rangle$ and an arbitrary observable $\mathcal{O}$, and outputs a random eigenvalue λ of $\mathcal{O}$ and a correpanding eigenket $|\lambda\rangle$ of $\mathcal{O}$ according to the laws of quantum mechanics.

Remark 1. *Please note that if the observable $\mathcal{O}$ has only one eigenvalue λ, then the output will simply be that eigenvalue λ and the original ket $|\psi\rangle$.*

TASK 2:

Next write a Maple program

$$QTomography(\psi, \epsilon)$$

that takes as input an **unkown** 2-qubit ket $|\psi\rangle$ and a positive real number ϵ, and then produces as output estimates $\widetilde{a}$ and $\widetilde{b}$ of the amplitudes a and b such that $|a - \widetilde{a}| < \epsilon$ and $\left|b - \widetilde{b}\right| < \epsilon$. [This program will make multiple calls to the subroutine Meas.]