Exercise 2
CMSC 691Q
Quantum Computation
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1 Problem 1.

Let ρ be a density operator giving the state of a qubit.

1) Prove that ρ can be written in the form

$$\rho = \frac{1}{2}\sigma_0 + \frac{1}{2}\overrightarrow{u} \cdot \overrightarrow{\sigma} \ ,$$

where $\overrightarrow{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the Pauli spin operators, where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

denotes the identity matrix, and where $\overrightarrow{u} = (u_1, u_2, u_3)$ is a vector in R^3 such that $\|\overrightarrow{u}\| \leq 1$.

2) Prove that, for every $\overrightarrow{u} \in R^3$ such that $\|\overrightarrow{u}\| \leq 1$,

$$\rho = \frac{1}{2}\sigma_0 + \frac{1}{2}\overrightarrow{u} \cdot \overrightarrow{\sigma}$$

is a density operator, i.e., that ρ is a positive semidefinite Hermtian operator of trace 1.

3) Prove that ρ is the state of a pure ensemble if and only if $\|\overrightarrow{u}\| = 1$. (**Hint:** ρ is a pure ensemble if and only if $\rho^2 = \rho$ if and only if $trace\left(\rho^2\right) = 1$.)

Remark 1 *You have just shown that all the possible states of a qubit can be naturally identified with the points of the 3-ball*

$$S_{Bloch} = \left\{ \overrightarrow{u} \in R^3 \mid \|\overrightarrow{u}\| \leq 1 \right\} \ ,$$

called the ***Bloch "sphere."*** *You have also just demonstrated that all possible pure ensemble qubit states can be naturally identified with the points of the surface of the Bloch "sphere," i.e., with the points of*

$$\partial S_{Bloch} = \left\{ \overrightarrow{u} \in R^3 \mid \|\overrightarrow{u}\| = 1 \right\} .$$

4) Let

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle$$

denote the state of an arbitrary qubit pure ensemble. (Without loss of generality assume that $|a|^2 + |b|^2 = 1$). Then the corresponding density operator is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\sigma_0 + \frac{1}{2}\overrightarrow{u}\cdot\overrightarrow{\sigma} .$$

Find expressions for the components of the vector $\overrightarrow{u}$ in terms of the parameters a and b.

5) Consider the following unitary transformations:

$$\begin{cases} U_X(\theta) & = & \exp\left(\frac{1}{2}\theta\sigma_1\right) \\ U_Y(\theta) & = & \exp\left(\frac{1}{2}\theta\sigma_2\right) \\ U_Z(\theta) & = & \exp\left(\frac{1}{2}\theta\sigma_3\right) \end{cases}$$

As θ varies from 0 to 2π, each of these unitary transformation can be viewd as a motion of the Bloch "sphere." Describe each of these motions. (**Hint:** Think about rotational motion.)

2 Problem 2.

Given that

1. Alice and Bob are separated by a great distance
2. Alice holds at her location a qubit labeled by A
3. Bob holds at his location a qubit labeled by B
4. Unknown to Alice and Bob, the state of the quantum system $\mathcal{Q}_{AB}$ consisting of qubits A and B is really given by the ket

$$|\psi_{AB}\rangle = \frac{4\,|0_A 0_B\rangle + 3\,|1_A 1_B\rangle}{5}$$

5. Alice holds at her location a third qubit labeled by C with a state (known by neither Alice nor Bob) given by the ket

$$|\psi_C\rangle = \frac{3\,|0_C\rangle + 2\sqrt{2}\,(1+i)\,|1_C\rangle}{5}$$

6. Alice and Bob erroneously assume that the state of $\mathcal{Q}_{AB}$ is the maximally entangled EPR state given by the ket

$$\frac{|0_A 0_B\rangle + |1_A 1_B\rangle}{\sqrt{2}}$$

7. Alice applies the *standard* [1] quantum teleportation protocol using $\mathcal{Q}_{AB}$ to teleport the state of qubit C to Bob.

After completion of the quantum teleportation protocol, exactly what is the qubit state received by Bob? How close is it to the original state of Qubit C? Please show your work, and explain in full.

References

[1] Gruska, Jozef, "**Quantum Computing**," McGraw-Hill (1999).

[2] Jozsa, Richard, **Quantum information and its properties**, in "**Introduction to Quantum Computation**," (edited by Lo, Hoi-Kwong, Sandu Popescu, and Tim Spiller), World Scientific (1998), pp 49 - 75.

[3] Neilsen, Michael A., and Isaac L. Chuang, "**Quantum Computation and Quantum Information**," Cambridge University Press, (2000).

[1] For a definition of "standard quantum teleportation protocol," please refer to [**?**, pages 26-28]. Please note that this protocol is slightly different from that given in [1, pages 251-256] and [2, pages 56-60]. But it is equivalent. Why?