Homework 1
CMSC 691Q
Quantum Computation
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1 Example Problem

Let $\mathcal{Q}$ be a quantum system with state given by the ket:

$$|\Psi\rangle = (|00\rangle + i\,|01\rangle - |11\rangle)/\sqrt{3}$$

What is the result of measuring $\mathcal{Q}$ with respect to the observable:

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}$$

Answer to Example Problem

The eigenkets and corresponding eigenvalues of $\mathcal{O}$ are:

Eigenvalue	Orthonormal Eigenkets(s)
$a_1 = \sqrt{2}$	$\|w_1\rangle = \frac{1}{2}\left(\sqrt{2}, 0, 1, i\right)^T$ $\|w_2\rangle = \frac{1}{2}\left(0, \sqrt{2}, -i, -1\right)^T$
$a_2 = -\sqrt{2}$	$\|w_3\rangle = \frac{1}{2}\left(i, 1, 0, \sqrt{2}\right)^T$ $\|w_4\rangle = \frac{1}{2}\left(-1, -i, \sqrt{2}, 0\right)^T$

Remark 1 *All eigenkets have been normalized to unit length. Please note that, while eigenkets corresponding to different eigenvalues are necessarily orthogonal, eigenkets corresponding to the same eigenvalue are not necessarily orthogals. So one must use the Gram Schmidt orthoganalization algorithm to obtain an orthoganal basis for each of the two 2-dimentional eigenspaces.*

Hence,

$$\mathcal{O} = a_1 \left(|w_1\rangle\langle w_1| + |w_2\rangle\langle w_2|\right) + a_2 \left(|w_3\rangle\langle w_3| + |w_4\rangle\langle w_4|\right)$$

is the spectral decomposition of the observable $\mathcal{O}$.

We can now express the state $|\psi\rangle$ of the quantum system in terms of the above eigenbasis determined by the observable $\mathcal{O}$.

$$|\psi\rangle = \left(\sum_{j=1}^{4} |w_j\rangle \langle w_j|\right) |\psi\rangle = \sum_{j=1}^{4} |w_j\rangle \langle w_j \mid \psi\rangle = \sum_{j=1}^{4} \alpha_j |w_j\rangle \ ,$$

where $\alpha_j = \langle w_j \mid \psi\rangle$. The values of the coefficients are:

$$\begin{cases} \alpha_1 & = & \langle w_1 \mid \psi\rangle & = & \frac{1}{6}\left(\sqrt{6} + i\sqrt{3}\right) \\ \alpha_2 & = & \langle w_2 \mid \psi\rangle & = & \frac{1}{6}\left(\sqrt{3} + i\sqrt{6}\right) \\ \alpha_3 & = & \langle w_3 \mid \psi\rangle & = & -\frac{1}{6}\sqrt{6} \\ \alpha_4 & = & \langle w_4 \mid \psi\rangle & = & -\frac{1}{3}\sqrt{3} \end{cases}$$

It follows that

$$\begin{aligned} |\psi\rangle &= \sum_{j=1}^{4} \alpha_j |w_j\rangle \\ &= \sqrt{|\alpha_1|^2 + |\alpha_2|^2} \left(\frac{\alpha_1 |w_1\rangle + \alpha_2 |w_2\rangle}{\sqrt{|\alpha_1|^2 + |\alpha_2|^2}} \right) + \sqrt{|\alpha_3|^2 + |\alpha_4|^2} \left(\frac{\alpha_3 |w_3\rangle + \alpha_4 |w_4\rangle}{\sqrt{|\alpha_3|^2 + |\alpha_4|^2}} \right) \end{aligned}$$

Hence, if $|\psi\rangle$ is measured with respect to the observable $\mathcal{O}$, we obtain:

Probability	Eigenvalue	Resulting State
$\lvert\alpha_1\rvert^2 + \lvert\alpha_2\rvert^2$	$\sqrt{2}$	$\frac{\alpha_1 \lvert w_1\rangle + \alpha_2 \lvert w_2\rangle}{\sqrt{\lvert\alpha_1\rvert^2 + \lvert\alpha_2\rvert^2}}$
$\lvert\alpha_3\rvert^2 + \lvert\alpha_4\rvert^2$	$-\sqrt{2}$	$\frac{\alpha_3 \lvert w_3\rangle + \alpha_4 \lvert w_4\rangle}{\sqrt{\lvert\alpha_3\rvert^2 + \lvert\alpha_4\rvert^2}}$

,

which when computed is found to be:

Probability	Eigenvalue	Resulting State
$\frac{1}{2}$	$\sqrt{2}$	$\left(\frac{\sqrt{6}+i\sqrt{3}}{6}, \frac{\sqrt{3}+i\sqrt{6}}{6}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{6}\right)^T$
$\frac{1}{2}$	$-\sqrt{2}$	$\left(\frac{\sqrt{6}-i\sqrt{3}}{6}, \frac{-\sqrt{3}+i\sqrt{6}}{6}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{6}\right)^T$

2 Problem 1.

Let $\mathcal{Q}$ be a quantum system with state given by the ket:

$$|\Psi\rangle = \left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)/4$$

What is the result of measuring $\mathcal{Q}$ with respect to the observable:

$$\mathcal{O} = \begin{pmatrix} 2 & 0 & 0 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -i & 0 & 0 & 2 \end{pmatrix}$$

3 Problem 2

Let $\mathcal{Q}$ be a quantum system with state given by the ket:

$$|\Psi\rangle = \left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)/4$$

What is the result of measuring $\mathcal{Q}$ with respect to the observable:

$$\mathcal{O} = \frac{1}{2}\begin{pmatrix} 5 & 0 & 0 & 3i \\ 0 & 5 & i & 0 \\ 0 & -i & 5 & 0 \\ -3i & 0 & 0 & 5 \end{pmatrix}$$

4 Example Problem

Let $\mathcal{Q}$ be a quantum system with state given by the density operator:

$$\rho = \begin{pmatrix} \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} \\ \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} \\ -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} & \frac{1}{4} \end{pmatrix}$$

What is the result of measuring $\mathcal{Q}$ with respect to the observable:

$$\mathcal{O} = \begin{pmatrix} 0 & -1 & -i & 0 \\ -1 & 0 & 0 & i \\ i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \end{pmatrix}$$

Answer to Example Problem

Remark 2 *Please note that, since* $Tr\left(\rho^2\right) = \frac{1}{3} < 1$, *it follows that* ρ *is a mixed ensemble.*

The eigenkets and corresponding eigenvalues of $\mathcal{O}$ are:

Eigenvalue	Orthonormal Eigenkets(s)	Projection Operator
$a_1 = 2$	$\|w_1\rangle = \frac{1}{2}(-1, 1, -i, -i)^T$	$P_1 = \|w_1\rangle\langle w_1\| = \frac{1}{4}\begin{pmatrix} 1 & -1 & -i & -i \\ -1 & 1 & i & i \\ i & -i & 1 & 1 \\ i & -i & 1 & 1 \end{pmatrix}$
$a_2 = -2$	$\|w_2\rangle = \frac{1}{2}(1, 1, -i, i)^T$	$P_2 = \|w_2\rangle\langle w_2\| = \frac{1}{4}\begin{pmatrix} 1 & 1 & i & -i \\ 1 & 1 & i & -i \\ -i & -i & 1 & -1 \\ i & i & -1 & 1 \end{pmatrix}$
$a_3 = 0$	$\|w_3\rangle = \frac{1}{2}\left(0, \sqrt{2}, i\sqrt{2}, 0\right)^T$ $\|w_4\rangle = \frac{1}{2}\left(i\sqrt{2}, 0, 0, \sqrt{2}\right)^T$	$P_3 = \|w_3\rangle\langle w_3\| + \|w_4\rangle\langle w_4\| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}$

Remark 3 *We have once again made sure that the eigenkets of the observable* $\mathcal{O}$ *form an orthonormal basis.*

Hence,

$$\mathcal{O} = a_1 P_1 + a_2 P_2 + a_3 P_3$$

is the spectral decomposition of the observable $\mathcal{O}$.

Remark 4 *As a check, we can verify that*

$$P_1 + P_2 + P_3 = I$$

is the identity operator, and also that $P_j^2 = P_j$ *for all* j.

Hence, if ρ is measured with respect to the observable $\mathcal{O}$, we obtain:

Probability	Eigenvalue	Resulting State
$p_1 = Tr(P_1\rho)$	2	$\rho_1 = \frac{P_1\rho P_1}{Tr(P_1\rho)}$
$p_2 = Tr(P_2\rho)$	-2	$\rho_2 = \frac{P_2\rho P_2}{Tr(P_2\rho)}$
$p_3 = Tr(P_3\rho)$	0	$\rho_3 = \frac{P_3\rho P_3}{Tr(P_3\rho)}$

,

which when computed is found to be:

Probability	Eigenvalue	Resulting State
$\frac{1}{6}$	2	$\frac{1}{4}\begin{pmatrix} 1 & -1 & -i & -i \\ -1 & 1 & i & i \\ i & -i & 1 & 1 \\ i & -i & 1 & 1 \end{pmatrix}$
$\frac{1}{6}$	-2	$\frac{1}{4}\begin{pmatrix} 1 & 1 & i & -i \\ 1 & 1 & i & -i \\ -i & -i & 1 & -1 \\ i & i & -1 & 1 \end{pmatrix}$
$\frac{2}{3}$	0	$\frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}$

Remark 5 *Since $\rho_1 = P_1$ and $\rho_2 = P_2$, it follows that $\rho_1^2 = \rho_1$ and $\rho_2^2 = \rho_2$. Hence, ρ_1 and ρ_2 are pure ensembles. On the other hand, since $\rho_3 = \frac{1}{2}P_3$, it follows that $\rho_3^2 = \frac{1}{2}\rho_3 \neq \rho_3$. Thus, ρ_3 is a mixed ensemble.*

5 Problem 3

Let $\mathcal{Q}$ be a quantum system with state given by the density operator:

$$\rho = \begin{pmatrix} \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} \\ \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{i}{12} & \frac{1}{4} & -\frac{i}{12} \\ -\frac{i}{12} & \frac{1}{12} & \frac{i}{12} & \frac{1}{4} \end{pmatrix}$$

What is the result of measuring $\mathcal{Q}$ with respect to the following observables:

a)

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}$$

b)

$$\mathcal{O} = \begin{pmatrix} 2 & 0 & 0 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -i & 0 & 0 & 2 \end{pmatrix}$$

c)

$$\mathcal{O} = \frac{1}{2}\begin{pmatrix} 5 & 0 & 0 & 3i \\ 0 & 5 & i & 0 \\ 0 & -i & 5 & 0 \\ -3i & 0 & 0 & 5 \end{pmatrix}$$