THE MACWILLIAMS AND PLESS IDENTITIES: A SUMMARY

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ABSTRACT. We give a brief summary of the MacWilliams and Pless Identies

1. Summary

Given the weight distribution of a linear code V, the MacWilliams and Pless identities provide a means of determining the weight distribution of the corresponding dual linear code V^{\perp} . We describe below these identities only for binary linear codes.

Theorem 1 (MacWilliams). Let V be a binary linear [n, k] code, and let V^{\perp} be the corresponding binary linear [n, n - k] orthogonal complement code. Let A_i and A_i^{\perp} denote the number of vectors of Hamming weight i in V and V^{\perp} , respectively Finally, let A(x) and $A^{\perp}(x)$ be the weight enumerator polynomials respectively defined by

$$A(x) = \sum_{i=0}^{n} A_i x^i$$
 and $A^{\perp}(x) = \sum_{i=0}^{n} A_i^{\perp} x^i$.

Then

$$2^{k} A^{\perp}(x) = (1+x)^{n} A\left(\frac{1-x}{1+x}\right)$$

Corollary 1 (Pless). Since

$$\left(x\frac{d}{dx}\right)^{j}A^{\perp}(x) = \sum_{i=0}^{n} i^{j}A_{i}^{\perp}x^{i} ,$$

we have

$$\left(x\frac{d}{dx}\right)^{j}A^{\perp}(x)\bigg|_{x=1} = \sum_{i=0}^{n} i^{j}A_{i}^{\perp} ,$$

Hence,

$$\sum_{i=0}^{n} A_{i}^{\perp} = 2^{n-k}$$

$$\sum_{i=0}^{n} i A_{i}^{\perp} = 2^{n-k-1} (n-A_{1})$$

$$\sum_{i=0}^{n} i^{2} A_{i}^{\perp} = 2^{n-k-2} [n (n+1) - 2nA_{1} + 2A_{2}]$$

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Example 1. Let V be the binary linear [5,3] code given by the generator matrix

Since the orthogonal complement V^{\perp} is of lower dimension 5-3=2, we will compute the weight enumerator $A^{\perp}(x)$ of V^{\perp} , and use the MacWilliams identity to find the weight energator of V, and hence the minimum distance d of V.

Since the generator matrix G is of the form G = (I | P), we know that the parity check matrix of V is given by

$$H = \left(\begin{array}{ccc} -P^T \mid I \end{array} \right) = \left(\begin{array}{cccc} 0 & 1 & 1 & \vdots & 1 & 0 \\ 1 & 0 & 1 & \vdots & 0 & 1 \end{array} \right)$$

Since the parity check matrix H of V is also the generator matrix of V^{\perp} , we can use H to enumerate the elements of V^{\perp} , i.e.,

$$V^{\perp} = \left\{ uH : u \in GF(2)^2 \right\} .$$

So with H, we can construct the following table enumerating the weights of V^{\perp} .

InfoWord	CodeWord	Weight
00	00000	0
01	01110	3
10	10101	3
11	11011	4

Let A_i^{\perp} be the number of elements of V^{\perp} of weight *i*. Then from the above table, we see that the values of A_i^{\perp} are:

i	A_i^{\perp}
0	1
1	0
2	0
3	2
4	1
5	0

Hence, the weight enumerator polynomial $A^{\perp}(x) = \sum_{i=0}^{5} A_i^{\perp} x^i$ is given by

$$A^{\perp}(x) = 1 \cdot x^{0} + 0 \cdot x^{1} + 0 \cdot x^{2} + 2 \cdot x^{3} + 1 \cdot x^{4} + 0 \cdot x^{5} = 1 + 2x^{3} + x^{4}$$

But from the MacWilliams identity, we know that

$$2^{2}A(x) = (1+x)^{5} A^{\perp} \left(\frac{1-x}{1+x}\right)$$

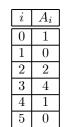
Thus,

$$4A(x) = (1+x)^5 \left[1 + 2\left(\frac{1-x}{1+x}\right)^3 + \left(\frac{1-x}{1+x}\right)^4 \right]$$
$$= (1+x)^5 + 2(1-x)^3(1+x)^2 + (1-x)^4(1+x)$$
$$= 4 + 8x^2 + 16x^3 + 4x^4$$

Thus,

$$A(x) = 1 + 2x^2 + 4x^3 + x^4$$

Hence, we have



where A_i denotes the number of elements of V of weight i. Thus, the minimum distance d of V is d = 2.

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