What Is the Braid Group B_n ???

The Braid Group & Beyond

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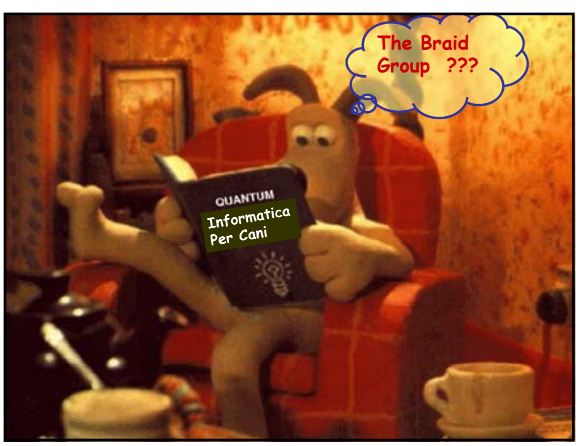
DARPA QuIST MSRI L-O-O-P

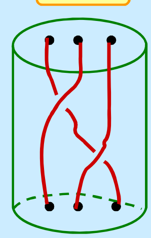
Why is the braid group important for Q Comp ?

- The representations of the Symmetric S_n are the basic building blocks for the representation of the unitary group U used in quantum mechanics,
- The braid group B_n "sits above" the symmetric group S_n, i.e., there is a natural epimorphism $B_n \downarrow S_n$
- Thus, new representations of the braid group B_n will give us new representations of the unitary group U, i.e., quantum gates
- **Claim**: These quantum gates can be implemented in quantum systems that are resistant to decoherence because of topological obstructions, e.g., in terms of the fractional quantum Hall effect, anyonic systems

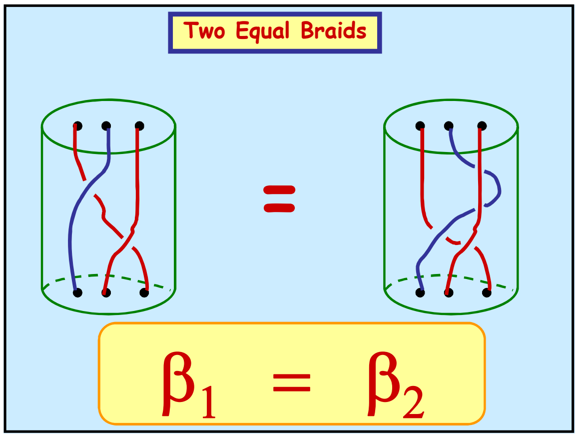


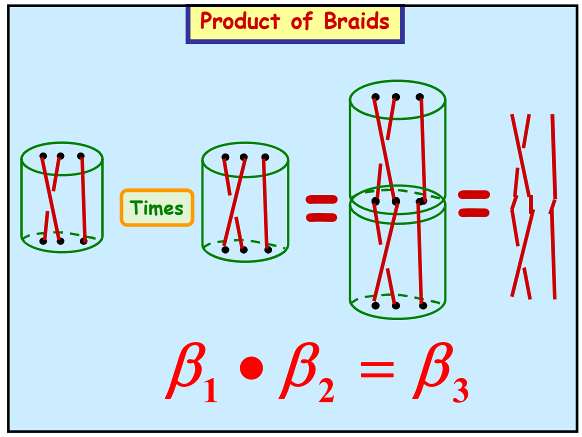
A Braid

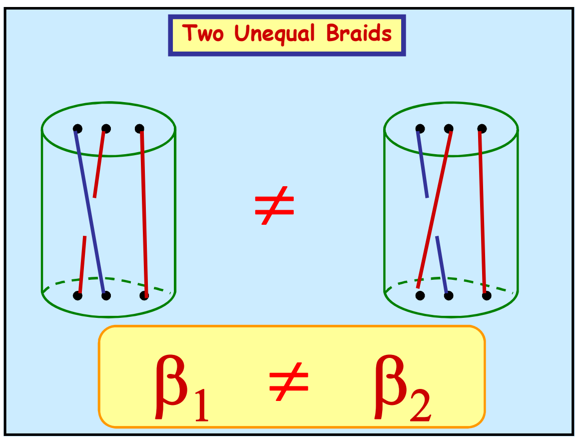
Hat Box

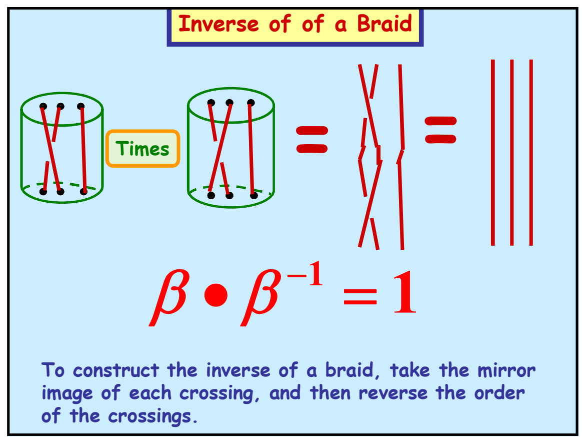


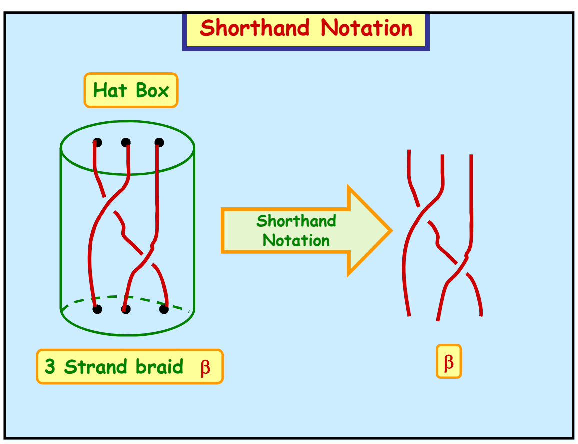
3 Strand braid β

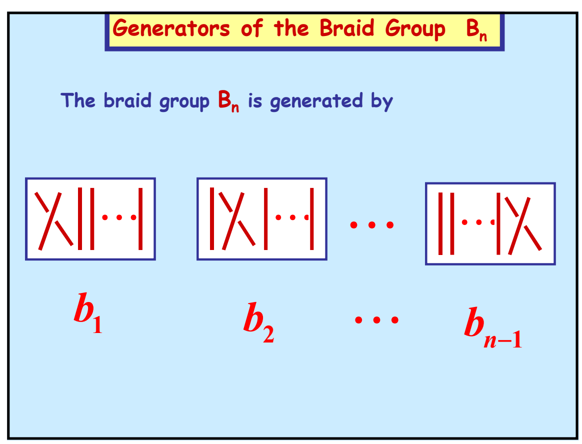

Two Equal Braids
$\beta_1 = \beta_2$


Product of Braids
Times
$\beta_1 \bullet \beta_2 = \beta_3$


Two Unequal Braids
$\beta_1 \neq \beta_2$


Inverse of of a Braid
Times
$\beta \bullet \beta^{-1} = 1$
To construct the inverse of a braid, take the mirror image of each crossing, and then reverse the order of the crossings.


Shorthand Notation
Hat Box
Shorthand Notation
3 Strand braid β
β


Generators of the Braid Group B_n
The braid group B_n is generated by
b_1
b_2
. . .
b_{n-1}

Relations Among the Generators of B_n

Reidemeister 3 Move

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}\ , 1 \le i < n$$

$$b_i b_j = b_j b_i \ , \text{for } |i-j| \ge 2$$

Skip to presentation

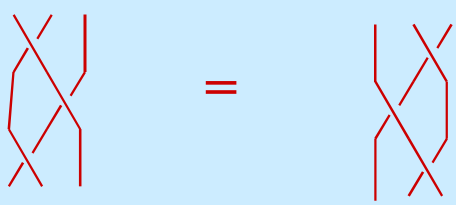
A Presentation of the Braid Group B_n

$$\left(b_1, b_2, \ldots, b_{n-1} : \begin{array}{l} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}\ , 1 \le i < n-1 \\ b_i b_j = b_j b_i\ , |i-j| > 1, 1 \le i, j < n-1 \end{array} \right)$$

Generators

Complete set of Relations

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}\ , 1 \le i < n$$



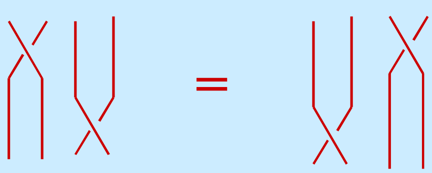
A Braid Is "Almost" a Permutation

$$B_n = \left(b_1, b_2, \ldots, b_{n-1} : \begin{array}{l} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}\ , 1 \le i < n-1 \\ b_i b_j = b_j b_i\ , |i-j| > 1, 1 \le i, j < n-1 \end{array} \right)$$

Natural Epimorphism

$$\bar{S}_n = \left(b_1, b_2, \ldots, b_{n-1} : \begin{array}{l} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}\ , \quad 1 \le i < n-1 \\ b_i^2 = 1, \quad 1 \le i < n-1 \\ b_i b_j = b_j b_i\ , |i-j| > 1, \quad 1 \le i, j < n-1 \end{array} \right)$$

$$b_i b_j = b_j b_i \ , \text{for } |i-j| \ge 2$$



Braids as Words

Every braid β in B_n can be written as a product of braid generators $b_1, b_2, \ldots, b_{n-1}$ and their inverses $b_1^{-1}, b_2^{-1}, \ldots, b_{n-1}^{-1}$

$$\beta = \prod_{i=1}^{L} b_{j(i)}^{\ \varepsilon(i)} = b_{j(1)}^{\ \varepsilon(1)} b_{j(2)}^{\ \varepsilon(2)} \cdots b_{j(L)}^{\ \varepsilon(L)}$$

Braid word w

where $\varepsilon(i) = \pm 1$

Anyons: A Very Brief Overview

Anyons are quantum systems that are confined to two dimensions. They were first proposed by Nobel Laureate F. Wilczek. See for example,

Wilczek, F., Fractional statistics and anyon superconductivity, World Scientific Press, (1990).

Anyons can used to explain the fractional quantum Hall effect

Anyons: A Very Brief Overview (Cont.)

Quantum Topology gives us the tools needed to find new unitary representations based on fusing and braiding

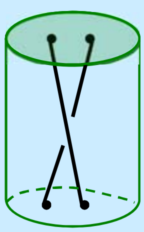
These new unitary transformations are created with an object called a unitary topological modular functor which we call simply an anyon model.

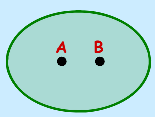
Recall: Q.M.= Qroup Rep. Theory

Another Perspective

A Braid Represents the Movement of n Holes in a Disc

This braiding can be used to represent Anyon exchanges





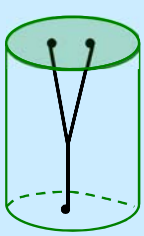
Anyonic braiding corresponds to a Unitary transformation

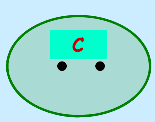
Recall: Q.M.= Qroup Rep. Theory

Knots from Braids

The Markov trace closure

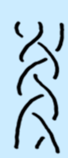
Anyons Can Also Fuse or Split



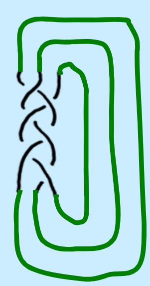


From Braids to Knots

The Markov Trace Closure of a Braid



Close Braid



Braid β

Closed Braid β^{tr}

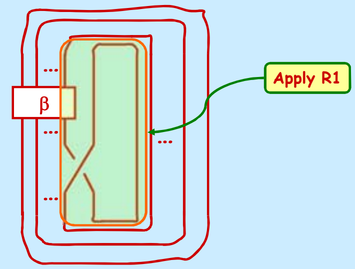
Can Every Knot Can Be Constructed from the Closure of Some braid?

Theorem (Alexander). Every knot is the closure of a braid.

Skip Markov Moves

The Markov 2 Move

M2 $\beta \mapsto \beta \bullet b_{n+1}^{\pm 1} \in B_{n+1}$



When Does the Closure of Two Braids Produce the Same Link ?

Theorem (Markov). Two braids β_1 and β_2 produce the same link under Markov trace braid closure iff there exists a finite sequence of Markov moves that transforms one braid into the other.

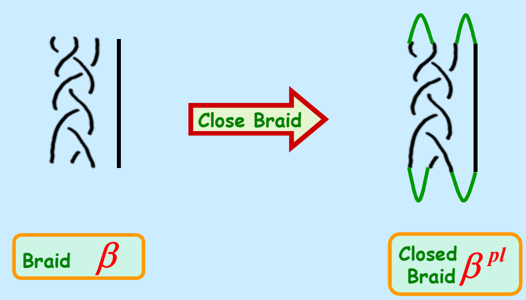
Definition. Let β be a braid in B_n. Then the Markov moves are defined as:

M1 $\beta \mapsto \beta' = b_i^{\pm 1} \bullet \beta \bullet b_i^{\mp 1}, \quad 1 \le i < n$

M2 $\beta \mapsto \beta \bullet b_{n+1}^{\pm 1} \in B_{n+1}$

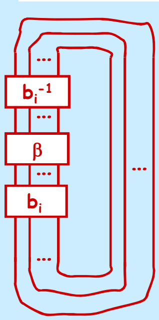
From Braids to Knots

The Plat Closure of a Braid



The Markov 1 Move

M1 $\beta \mapsto \beta' = b_i^{\pm 1} \bullet \beta \bullet b_i^{\mp 1}, \quad 1 \le i < n$



The Aharonov-Freedman-Jones-Kitaev-Landau (AFJKL) Algorithm

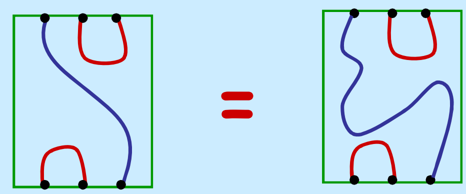
Strategy for Computing Jones Polynomial

- We use the Jones representation

$$\rho_A : B_n \to TL_n(d)$$

from the braid group B_n to the Temperley-Lieb algebra $TL_n(d)$, where d is an indeterminate complex number, and where A is a complex number defined by $d=-A^2-A^{-2}$.

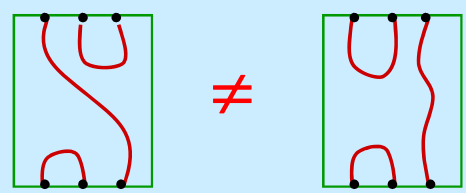
Two Equal Elements of $TL_n(d)$



$X_1 = X_2$

What is the Temperley-Lieb algebra $TL_n(d)$???

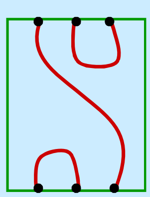
Two Unequal Elements of $TL_n(d)$



$X_1 \neq X_2$

Diagrammatic Representation of $TL_n(d)$

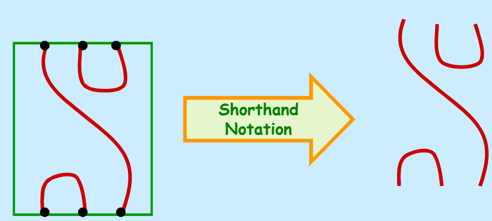
Rectangle



An element X of $TL_n(d)$

Shorthand Notation

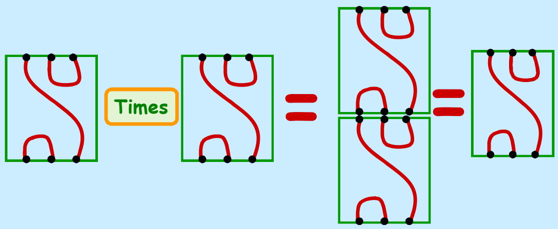
Rectangle



X

X

Product



$$X_1 \bullet X_2 = X_3$$

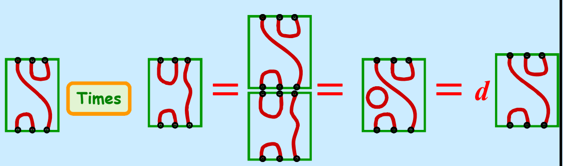
Relations Among the Generators of $TL_n(d)$

$$E_i E_j = E_j E_i \text{ , for } |i-j| \geq 2$$

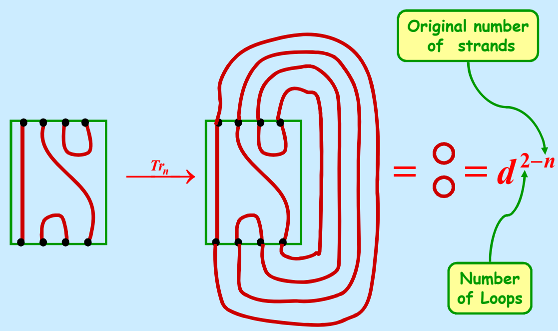
$$E_i E_{i\pm1} E_i = E_i \text{ , } 1 \leq i < n$$

$$E_i^{\,2} = dE_i \text{ , } 1 \leq i < n$$

Product

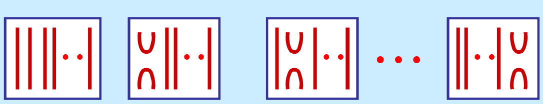


The Markov Trace $Tr_n : TL_n(d) \rightarrow C$



Generators of the Temperley-Lieb Algebra $TL_n(d)$

The Temperley-Lieb algebra $TL_n(d)$ is generated by



$$1 \qquad E_1 \qquad E_2 \qquad \cdots \qquad E_{n-1}$$

Properties of the Markov Trace

The Markov trace is the unique trace satisfying the following conditions:

- $Tr_n(1) = 1$
- $Tr_n(XY) = Tr_n(YX)$
- $X \in TL_{n-1}(d) \Rightarrow Tr_{n-1}(XE_n) = \frac{1}{d} Tr_n(X)$

How to Compute the Jones Polynomial