In[1]:=
```
(* EPR = (|01>-|10>)/√2 *)
EPR = (1/√2) * {{0}, {1}, {-1}, {0}}
```

Out[1]=
$$\left\{\{0\}, \left\{\frac{1}{\sqrt{2}}\right\}, \left\{-\frac{1}{\sqrt{2}}\right\}, \{0\}\right\}$$

In[2]:=
```
(* Ψ = (|01>-|10>)/√2 (|01>-|10>)/√2 = (|0101>-|0110>-|1001>+|1010>)/2 *)
Ψ = KroneckerProduct[EPR, EPR]
MatrixForm[Ψ]
```

Out[2]=
$$\left\{\{0\}, \{0\}, \{0\}, \{0\}, \{0\}, \left\{\frac{1}{2}\right\}, \left\{-\frac{1}{2}\right\}, \{0\}, \{0\}, \left\{-\frac{1}{2}\right\}, \left\{\frac{1}{2}\right\}, \{0\}, \{0\}, \{0\}, \{0\}, \{0\}\right\}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In[4]:=
```
(* Bell Basis *)
Φ1 = (1/√2) * {{1}, {0}, {0}, {1}}; (* Φ1 = (|00>+|11>)/√2 *) MatrixForm[Φ1]
Φ2 = (1/√2) * {{0}, {1}, {1}, {0}}; (* Φ1 = (|001>+|10>)/√2 *) MatrixForm[Φ2]
Φ3 = (1/√2) * {{0}, {1}, {-1}, {0}}; (* Φ1 = (|001>-|10>)/√2 *) MatrixForm[Φ3]
Φ4 = (1/√2) * {{1}, {0}, {0}, {-1}}; (* Φ1 = (|00>-|11>)/√2 *) MatrixForm[Φ1]
```

Out[4]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Out[5]//MatrixForm=

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Out[6]//MatrixForm=

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Out[7]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

In[8]:=
```
(* Bell Projectors *)
P1 = Φ1.Φ1†; MatrixForm[P1]
P2 = Φ2.Φ2†; MatrixForm[P2]
P3 = Φ3.Φ3†; MatrixForm[P3]
P4 = Φ4.Φ4†; MatrixForm[P4]
```

Out[8]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Out[9]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[10]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[11]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

In[12]:=
```
(* Testing that these are actually projectors *)
P1.P1 == P1
P2.P2 == P2
P3.P3 == P3
P4.P4 == P4
```

Out[12]= True

Out[13]= True

Out[14]= True

Out[15]= True

In[16]:=
```
(* Testing for orthogonality *)

P1.P2
```

Out[16]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

In[17]:=
```
(* Bell Measurement of middle 2 qubits of Ψ *)
```

In[18]:= **I2 =** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

```
PP1 = KroneckerProduct[KroneckerProduct[I2, P1], I2]; MatrixForm[PP1]
PP2 = KroneckerProduct[KroneckerProduct[I2, P2], I2]; MatrixForm[PP2]
PP3 = KroneckerProduct[KroneckerProduct[I2, P3], I2]; MatrixForm[PP3]
PP4 = KroneckerProduct[KroneckerProduct[I2, P4], I2]; MatrixForm[PP4]
```

Out[18]= {{1, 0}, {0, 1}}

Out[19]//MatrixForm=

$$\begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}$$

Out[20]//MatrixForm=

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Out[21]//MatrixForm=

$$
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0\\
0&0&-\frac{1}{2}&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-\frac{1}{2}&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0\\
0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&\frac{1}{2}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&\frac{1}{2}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

Out[22]//MatrixForm=

$$
\begin{pmatrix}
\frac{1}{2}&0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0\\
0&\frac{1}{2}&0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-\frac{1}{2}&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0\\
0&-\frac{1}{2}&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&-\frac{1}{2}&0\\
0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&-\frac{1}{2}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0&\frac{1}{2}&0\\
0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0&\frac{1}{2}
\end{pmatrix}
$$

In[23]:=
```
{{Prob1}} = Ψ†.PP1.Ψ; Prob1
Ψ1 = PP1.Ψ / √Prob1; MatrixForm[Ψ1]
```

Out[23]=
$$\frac{1}{4}$$

Out[24]//MatrixForm=
$$\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

In[25]:=
$$\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$
```
(* = -(|0000>+|0110>+|1001>+|1111>)/2 *)
```

Out[25]=
$$\left\{\left\{-\frac{1}{2}\right\}, \{0\}, \{0\}, \{0\}, \{0\}, \{0\}, \left\{-\frac{1}{2}\right\}, \{0\}, \{0\}, \left\{-\frac{1}{2}\right\}, \{0\}, \{0\}, \{0\}, \{0\}, \{0\}, \left\{-\frac{1}{2}\right\}\right\}$$

In[26]:=
```
{{Prob2}} = Ψ†.PP2.Ψ; Prob2
Ψ2 = PP2.Ψ / √Prob2; MatrixForm[Ψ2]
```

Out[26]= $\frac{1}{4}$

Out[27]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In[28]:=
$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ (* = $\frac{|0011\rangle+|0101\rangle+|1010\rangle+|1100\rangle}{2}$ *)

Out[28]= $\left\{\{0\}, \{0\}, \{0\}, \left\{\frac{1}{2}\right\}, \{0\}, \left\{\frac{1}{2}\right\}, \{0\}, \{0\}, \{0\}, \{0\}, \left\{\frac{1}{2}\right\}, \{0\}, \left\{\frac{1}{2}\right\}, \{0\}, \{0\}, \{0\}\right\}$

In[29]:=
```
{{Prob3}} = Ψ†.PP3.Ψ; Prob3
Ψ3 = PP3.Ψ / √Prob3; MatrixForm[Ψ3]
```

Out[29]= $\frac{1}{4}$

Out[30]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (* = \frac{-|0011\rangle+|0101\rangle+|1010\rangle-|1100\rangle}{2} *)$$

Out[31]= $\left\{\{0\}, \{0\}, \{0\}, \left\{-\frac{1}{2}\right\}, \{0\}, \left\{\frac{1}{2}\right\}, \{0\}, \{0\}, \{0\}, \{0\}, \left\{\frac{1}{2}\right\}, \{0\}, \left\{-\frac{1}{2}\right\}, \{0\}, \{0\}, \{0\}\right\}$

In[32]:=
```
{{Prob4}} = Ψ†.PP4.Ψ; Prob4
Ψ4 = PP4.Ψ / √Prob4; MatrixForm[Ψ4]
```

Out[32]= $\frac{1}{4}$

Out[33]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

In[34]:=
$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (* \; = \; \frac{|0000\rangle - |0110\rangle - |1001\rangle + |1111\rangle}{2} \; *)$$

```
Ψ3 = PP3.Ψ / √Prob3; MatrixForm[Ψ3]
```

Out[34]= $\left\{\left\{\frac{1}{2}\right\}, \{0\}, \{0\}, \{0\}, \{0\}, \{0\}, \left\{-\frac{1}{2}\right\}, \{0\}, \{0\}, \left\{-\frac{1}{2}\right\}, \{0\}, \{0\}, \{0\}, \{0\}, \{0\}, \left\{\frac{1}{2}\right\}\right\}$

Out[35]//MatrixForm=
$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In[36]:=

0	Resulting State	Probability
Measure wrt P1	Result1 = $-\frac{|0000\rangle + |0110\rangle + |1001\rangle + |1111\rangle}{2}$	$\frac{1}{4}$
Measure wrt P2	Result2 = $\frac{|0011\rangle + |0101\rangle + |1010\rangle + |1100\rangle}{2}$	$\frac{1}{4}$
Measure wrt P3	Result3 = $\frac{-|0011\rangle + |0101\rangle + |1010\rangle - |1100\rangle}{2}$	$\frac{1}{4}$
Measure wrt P4	Result4 = $\frac{|0000\rangle - |0110\rangle - |1001\rangle + |1111\rangle}{2}$	$\frac{1}{4}$

The renaming of the qubits given by the permutation $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$ **=**

(0 123 000 000) induces the permutation of the basis states given by

| b3, b2, b1, b0 > -- > | b0, b3, b2, b1 >

This induced permutation is Perm =

(0) (1, 2, 48) (5, 10) (6, 12, 9, 3) (7, 14, 13, 11) (15),

where, for example, 9 stands for 1001. After applying Prm, we have :

0	Resulting State	Probability
Measure wrt P1	$\text{Result1}' = -\frac{\lvert 0000\rangle+\lvert 1100\rangle+\lvert 0011\rangle+\lvert 1111\rangle}{2} = -\frac{\lvert 00\rangle+\lvert 11\rangle}{\sqrt{2}} \cdot \frac{\lvert 00\rangle+\lvert 11\rangle}{\sqrt{2}}$	$\frac{1}{4}$
Measure wrt P2	$\text{Result2}' = \frac{\lvert 0110\rangle+\lvert 1010\rangle+\lvert 0101\rangle+\lvert 1001\rangle}{2} = \frac{\lvert 01\rangle+\lvert 10\rangle}{\sqrt{2}} \cdot \frac{\lvert 01\rangle+\lvert 10\rangle}{\sqrt{2}}$	$\frac{1}{4}$
Measure wrt P3	$\text{Result3}' = \frac{-\lvert 0110\rangle+\lvert 1010\rangle+\lvert 0101\rangle-\lvert 1001\rangle}{2} = \frac{\lvert 01\rangle-\lvert 10\rangle}{\sqrt{2}} \cdot \frac{\lvert 01\rangle-\lvert 10\rangle}{\sqrt{2}}$	$\frac{1}{4}$
Measure wrt P4	$\text{Result4}' = \frac{\lvert 0000\rangle-\lvert 1100\rangle-\lvert 0011\rangle+\lvert 1111\rangle}{2} = \frac{\lvert 00\rangle-\lvert 11\rangle}{\sqrt{2}} \cdot \frac{\lvert 00\rangle-\lvert 11\rangle}{\sqrt{2}}$	$\frac{1}{4}$

Thus, before the, qubits 0 & 1 and qubits 2 & 3 respectivly form EPR pairs.

But after the measurement, qubits 0 & 2 and qubits 1 & 3 respectively form EPR pairs.

Hence we have "swapped" the entanglement.

In[36]:=

In[37]:=