A Substitution Method Solution to Exercise 4.4-1 on page 92

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A substitution method solution for Exercise 4.4-1 (page 92) of the text is given below:

Proposition 1 *Let $T(n)$ be the recursion defined by*

$$T(n) = \begin{cases} 1 & \textit{if } \ n = 1 \\ \\ 3T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n & \textit{if } \ n > 1 \end{cases}$$

The

$$T(n) = O\left(n^{\lg 3}\right)$$

Proof. Inductive Hyothesis Step:

Assume for a fixed but arbitrary integer $k > 1$ that

$$T(m) \leq cm^{\lg 3} - bm - d$$

for all m such that $1 \leq m < k$.

Inductive Step:

[We now wish to use the iductive hypothesis to show that $T(k) \leq ck^{\lg 3} - bk - d$.]

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\begin{aligned}
T(k) &= 3T\left(\left\lfloor \tfrac{k}{2} \right\rfloor\right) + k && \text{Reason: Def. of } T(n) \\
&\leq 3\left(c\left(\left\lfloor \tfrac{k}{2} \right\rfloor\right)^{\lg 3} - b\left\lfloor \tfrac{k}{2} \right\rfloor - d\right) + k && \text{Reason: Ind. Hyp.} \\
&\leq 3\left(c\left(\tfrac{k}{2}\right)^{\lg 3} - b\left(\tfrac{k}{2} - 1\right) - d\right) + k && \text{Reason: B.A.} \\
&\leq 3c\left(\tfrac{k^{\lg 3}}{2^{\lg 3}}\right) - \tfrac{3bk}{2} + 3b - 3d + k && \text{Reason: B.A.} \\
&\leq ck^{\lg 3} - \tfrac{3}{2}bk + 3b - 3d + k && \text{Reason: } 2^{\lg 3} = 3 \text{ \& B.A.} \\
&\leq ck^{\lg 3} - \left(\tfrac{3}{2}b - 1\right)k - 3(d - b) && \text{Reason: B.A.}
\end{aligned}
$$

Thus, $T(k) \leq ck^{\lg 3} - bk - d$ provided that

$$
\begin{cases}
-\left(\tfrac{3}{2}b - 1\right) \leq -b \\
-3(d - b) \leq -d
\end{cases}
$$

This last system of inequalities is equivalent to

$$
\begin{cases}
b \geq 2 \\
d \geq \tfrac{3}{2}b
\end{cases}
$$

Hence the induction step goes through, if we choose $b = 2$ and $d = 3$.

Basis Step:
$1 = T(1) \leq c \cdot 1^{\lg 3} - 2 \cdot 1 - 3$ is true, provided we choose $c \geq 6$.