Homework 1
CMSC 643
Quantum Computation
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1 Example Problem

Let $\mathcal{Q}$ be a quantum system with state given by the ket:

$$|\Psi\rangle = \left(|00\rangle + i\,|01\rangle - |11\rangle\right)/\sqrt{3}$$

What is the result of measuring $\mathcal{Q}$ with respect to the observable:

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}$$

Answer to Example Problem

The eigenkets and corresponding eigenvalues of $\mathcal{O}$ are:

Eigenvalue	Orthonormal Eigenkets(s)
$a_1 = \sqrt{2}$	$\|w_1\rangle = \frac{1}{2}\left(\sqrt{2}, 0, 1, i\right)^T$ $\|w_2\rangle = \frac{1}{2}\left(0, \sqrt{2}, -i, -1\right)^T$
$a_2 = -\sqrt{2}$	$\|w_3\rangle = \frac{1}{2}\left(i, 1, 0, \sqrt{2}\right)^T$ $\|w_4\rangle = \frac{1}{2}\left(-1, -i, \sqrt{2}, 0\right)^T$

Remark 1 *All eigenkets have been normalized to unit length. Please note that, while eigenkets corresponding to different eigenvalues are necessarily orthogonal, eigenkets corresponding to the same eigenvalue are not necessarily orthogals. So one must use the Gram Schmidt orthoganalization algorithm to obtain an orthoganal basis for each of the two 2-dimentional eigenspaces.*

Hence,

$$\mathcal{O} = a_1\left(|w_1\rangle\langle w_1| + |w_2\rangle\langle w_2|\right) + a_2\left(|w_3\rangle\langle w_3| + |w_4\rangle\langle w_4|\right)$$

is the spectral decomposition of the observable $\mathcal{O}$.

We can now express the state $|\psi\rangle$ of the quantum system in terms of the above eigenbasis determined by the observable $\mathcal{O}$.

$$|\psi\rangle = \left(\sum_{j=1}^{4} |w_j\rangle \langle w_j|\right) |\psi\rangle = \sum_{j=1}^{4} |w_j\rangle \langle w_j \mid \psi\rangle = \sum_{j=1}^{4} \alpha_j |w_j\rangle \ ,$$

where $\alpha_j = \langle w_j \mid \psi\rangle$. The values of the coefficients are:

$$\begin{cases} \alpha_1 &= \langle w_1 \mid \psi\rangle &= \frac{1}{6}\left(\sqrt{6} + i\sqrt{3}\right) \\ \alpha_2 &= \langle w_2 \mid \psi\rangle &= \frac{1}{6}\left(\sqrt{3} + i\sqrt{6}\right) \\ \alpha_3 &= \langle w_3 \mid \psi\rangle &= -\frac{1}{6}\sqrt{6} \\ \alpha_4 &= \langle w_4 \mid \psi\rangle &= -\frac{1}{3}\sqrt{3} \end{cases}$$

It follows that

$$\begin{aligned} |\psi\rangle &= \sum_{j=1}^{4} \alpha_j |w_j\rangle \\ &= \sqrt{|\alpha_1|^2 + |\alpha_2|^2}\left(\frac{\alpha_1 |w_1\rangle + \alpha_2 |w_2\rangle}{\sqrt{|\alpha_1|^2 + |\alpha_2|^2}}\right) + \sqrt{|\alpha_3|^2 + |\alpha_4|^2}\left(\frac{\alpha_3 |w_3\rangle + \alpha_4 |w_4\rangle}{\sqrt{|\alpha_3|^2 + |\alpha_4|^2}}\right) \end{aligned}$$

Hence, if $|\psi\rangle$ is measured with respect to the observable $\mathcal{O}$, we obtain:

Probability	Eigenvalue	Resulting State
$\|\alpha_1\|^2 + \|\alpha_2\|^2$	$\sqrt{2}$	$\frac{\alpha_1 \|w_1\rangle + \alpha_2 \|w_2\rangle}{\sqrt{\|\alpha_1\|^2 + \|\alpha_2\|^2}}$
$\|\alpha_3\|^2 + \|\alpha_4\|^2$	$-\sqrt{2}$	$\frac{\alpha_3 \|w_3\rangle + \alpha_4 \|w_4\rangle}{\sqrt{\|\alpha_3\|^2 + \|\alpha_4\|^2}}$

,

which when computed is found to be:

Probability	Eigenvalue	Resulting State
$\frac{1}{2}$	$\sqrt{2}$	$\left(\frac{\sqrt{6}+i\sqrt{3}}{6}, \frac{\sqrt{3}+i\sqrt{6}}{6}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{6}\right)^T$
$\frac{1}{2}$	$-\sqrt{2}$	$\left(\frac{\sqrt{6}-i\sqrt{3}}{6}, \frac{-\sqrt{3}+i\sqrt{6}}{6}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{6}\right)^T$

2 Exercise 1.1

Let $\mathcal{Q}$ be a quantum system with state given by the ket:

$$|\Psi\rangle = \left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)/2$$

What is the result of measuring $\mathcal{Q}$ with respect to the observable:

$$\mathcal{O} = \begin{pmatrix} 2 & 0 & 0 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -i & 0 & 0 & 2 \end{pmatrix}$$

3 Exercise 1.2

Let $\mathcal{Q}$ be a quantum system with state given by the ket:

$$|\Psi\rangle = \left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)/2$$

What is the result of measuring $\mathcal{Q}$ with respect to the observable:

$$\mathcal{O} = \frac{1}{2}\begin{pmatrix} 5 & 0 & 0 & 3i \\ 0 & 5 & i & 0 \\ 0 & -i & 5 & 0 \\ -3i & 0 & 0 & 5 \end{pmatrix}$$