

**CMSC 203 SECTION 0301 FALL 2007**  
**ANSWER TO HOMEWORK 6, NUMBER 28, PAGE 234**  
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1. TWO WAYS TO ANSWER HOMEWORK 6, EXERCISE 4.3, NUMBER 28, PAGE 234

**Statement of the problem:**

*Prove that for all integers  $n \geq 1$ ,*

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots = \frac{1+3+\cdots+(2n-1)}{(2n+1)+\cdots+(4n-1)}$$

**1.1. Method 1: Proof using formula for arithmetic series and no math induction.**

**Proposition 1.**  $\sum_{j=1}^n (2j - 1) = n^2$  for  $n \geq 1$

*Proof.*

$$\sum_{j=1}^n (2j - 1) = 2 \sum_{j=1}^n j - \sum_{j=1}^n 1 = 2 \left( \frac{n(n+1)}{2} \right) - n = n(n+1) - n = n^2$$

□

**Corollary 1.** For  $n \geq 1$ ,

$$\frac{\sum_{j=1}^n (2j - 1)}{\sum_{j=n+1}^{2n} (2j - 1)} = \frac{1}{3}$$

*Proof.* By the above proposition

$$\sum_{j=1}^n (2j - 1) = n^2 \quad \text{and} \quad \sum_{j=1}^{2n} (2j - 1) = 4n^2$$

Hence,

$$\sum_{j=n+1}^{2n} (2j - 1) = \sum_{j=1}^{2n} (2j - 1) - \sum_{j=1}^n (2j - 1) = 4n^2 - n^2 = 3n^2$$

Therefore,

$$\frac{\sum_{j=1}^n (2j - 1)}{\sum_{j=n+1}^{2n} (2j - 1)} = \frac{n^2}{3n^2} = \frac{1}{3}$$

□

### 1.2. Method 2: Proof using math induction and summation notation.

**Proposition 2.** For  $n \geq 1$ ,

$$P(n) : \frac{\sum_{j=1}^n (2j - 1)}{\sum_{j=n+1}^{2n} (2j - 1)} = \frac{1}{3}$$

*Proof (By Math Induction):*

**Basis Step:**  $P(1)$  is true, for:

$$\frac{\sum_{j=1}^1 (2j - 1)}{\sum_{j=1+1}^{2 \cdot 1} (2j - 1)} = \frac{2 \cdot 1 - 1}{2 \cdot 2 - 1} = \frac{1}{3}$$

**Induction Hypothesis:** Assume for a fixed but arbitrary integer  $k \geq 1$  that  $P(k)$  is true, i.e., that

$$\frac{\sum_{j=1}^k (2j - 1)}{\sum_{j=k+1}^{2k} (2j - 1)} = \frac{1}{3}$$

**Inductive Step:** We now use the inductive hypothesis to prove that  $P(k + 1)$  is true.

$$\begin{aligned} \frac{\sum_{j=1}^{k+1} (2j - 1)}{\sum_{j=(k+1)+1}^{2(k+1)} (2j - 1)} &= \frac{\left(\sum_{j=1}^k (2j - 1)\right) + [2(k + 1) - 1]}{\left(\sum_{j=k+1}^{2k} (2j - 1)\right) - [2(k + 1) - 1] + [2(2k + 1) - 1] + [2(2k + 2) - 1]} \\ &= \frac{\left(\sum_{j=1}^k (2j - 1)\right) + 2k + 1}{\left(\sum_{j=k+1}^{2k} (2j - 1)\right) + 3(2k + 1)} \end{aligned}$$

But by the induction hypothesis,

$$\frac{\sum_{j=1}^k (2j - 1)}{\sum_{j=k+1}^{2k} (2j - 1)} = \frac{1}{3}$$

So it follows that

$$\sum_{j=k+1}^{2k} (2j - 1) = 3 \left( \sum_{j=k+1}^{2k} (2j - 1) \right)$$

Thus,

$$\begin{aligned}
 \frac{\sum_{j=1}^{k+1} (2j-1)}{\sum_{j=(k+1)+1}^{2(k+1)} (2j-1)} &= \frac{\left(\sum_{j=1}^k (2j-1)\right) + 2k+1}{\left(\sum_{j=k+1}^{2k} (2j-1)\right) + 3(2k+1)} \\
 &= \frac{\left(\sum_{j=1}^k (2j-1)\right) + 2k+1}{3\left(\sum_{j=k+1}^{2k} (2j-1)\right) + 3(2k+1)} \\
 &= \frac{1}{3} \cdot \frac{\left(\sum_{j=1}^k (2j-1)\right) + 2k+1}{\left(\sum_{j=k+1}^{2k} (2j-1)\right) + 2k+1} = \frac{1}{3}
 \end{aligned}$$

**Magic Wand Step:** By the principle of mathematical induction  $P(n)$  is true for all integers  $n \geq 1$ , i.e.,

$$\frac{\sum_{j=1}^n (2j-1)}{\sum_{j=n+1}^{2n} (2j-1)} = \frac{1}{3} \text{ for } n \geq 1$$

□

**Remark 1.** '□' denotes Q.E.D.