Manipulating the Entanglement of One Copy of a Two-Particle Pure Entangled State

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ABSTRACT. One copy of a general bipartite pure entangled quantum state is considered. We show how it is possible to manipulate the entanglement by local operations and classical communication in order to produce to produce maximally entangled states. It is shown that this problem is equivalent to a certain colouring problem.

There has been much work recently concerning the manipulation of the entanglement of one copy of a pure bipartite entangled state by local operations and classical communication (LOCC). Lo and Popescu [**1**] obtained some important results in 1997 which were taken up by other researchers more recently. Nielsen [**2**] obtained necessary and sufficient conditions for it to be possible to transform one state by LOCC into another. Vidal [**3**] developed entanglement monotones (quantities which cannot increase under LOCC transformations) and studied probabilistic transformations of one state to another. Jonathan and Plenio [**4**] showed that some transformations which cannot be performed directly can nevertheless be performed if the parties are allowed to borrow entanglement which they later give back.

In this paper we will consider the problem of how to manipulate the entanglement of a general bipartite pure state by local operations and classical communication (LOCC) in order to produce maximally entangled states. This problem was originally studied in a limited context by Lo and Popescu [**1**]. In Hardy [**5**] and Jonathan and Plenio [**6**] the problem was solved in full generality. The purpose of this paper is to present the main points in [**5**]. The details of some of the more involved proofs are not given here and for those the reader is referred to [**5**].

We start with two systems A and B prepared in a general pure state $|\psi\rangle$. It follows from the Schmidt decomposition theorem that we can write

$$|\psi\rangle = \sum_{j=1}^{I} \sqrt{\lambda_j}|j\rangle_A|j\rangle_B \tag{1}$$

where by appropriate labeling we choose $\lambda_j \geq \lambda_{j+1}$ and where the states $|j\rangle_{A,B}$ are orthonormal. We have $\sum_i \lambda_i = 1$ by normalisation. Let system A be in Alice's possession and let system B be in Bob's possession. We will assume that Alice and Bob are allowed to do anything which does not involve their sending quantum systems to one another (for if they could do this then they could generate new entanglement between themselves - for example, Alice could send one particle of an entangled pair to Bob and keep the other particle in her own hands). This means that Alice and Bob can do anything locally that they want, in particular they can make measurements, introduce

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new local quantum systems, and discard quantum systems. In addition, they can communicate classically with one another, for example by talking down a telephone wire. Their task here is to obtain states like

$$|\varphi_m\rangle = \frac{1}{\sqrt{m}} \sum_{k=1}^{m} |k\rangle_A |k\rangle_B \tag{2}$$

We will call this an m-state. This is a maximally entangled state involving m levels of systems A and B. We say it is maximally entangled because modulus of the amplitude of each term when written in this Schmidt form is equal. The bigger m is the more entanglement there is in an m-state. The usual measure that is used is that one m-state is equivalent to $\log_2 m$ 2-states. Because of their simplicity, 2-states, such as the state

$$\frac{1}{\sqrt{2}}(|1\rangle_A|1\rangle_B + |2\rangle_A|2\rangle_B),$$

are a useful unit for measuring quantum entanglement. The $\log_2 m$ relation makes most sense when $m = 2^r$ for integer r. Consider $m = 4$. Since $\log_2 4 = 2$ a 4-state should be equivalent to two 2-states. We can see this is true. Take two 2-states

$$\frac{1}{2}(|1\rangle_{A'}|1\rangle_{B'} + |2\rangle_{A'}|2\rangle_{B'})(|1\rangle_{A''}|1\rangle_{B''} + |2\rangle_{A''}|2\rangle_{B''})$$

We can expand this:

$$\frac{1}{\sqrt{4}}(|11\rangle_{A'A''}|11\rangle_{B'B''} + |12\rangle_{A'A''}|12\rangle_{B'B''}|21\rangle_{A'A''}|21\rangle_{B'B''} + |22\rangle_{A'A''}|22\rangle_{B'B''})$$

where we have used the notation $|1\rangle_{A'}|1\rangle_{A''} = |11\rangle_{A'A''}$. Now we can think of the system $A'A''$ as a single system A and similarly for B. Then we can rename the states in this expansion as follows

$$\begin{aligned}
|11\rangle_{A'A''} &= |1\rangle_A \\
|12\rangle_{A'A''} &= |2\rangle_A \\
|21\rangle_{A'A''} &= |3\rangle_A \\
|22\rangle_{A'A''} &= |4\rangle_A
\end{aligned}$$

and similarly for B. Hence the above state becomes

$$\frac{1}{\sqrt{4}} \sum_{k=1}^{4} |k\rangle_A |k\rangle_B$$

and this is just a 4-state. It is clear that the same can be done for any other $m = 2^r$. This proves the $\log_2 m$ relation for values of m which are powers of 2. Although this relation works strictly only for these values, it is reasonable to use this as a measure for other values of m also (especially when we consider that if Alice and Bob share an m-state and an m'-state then this is equivalent to their sharing a state with mm' terms, namely an mm'-state and therefore the measure is additive because $\log_2 m + \log_2 m' = \log_2 mm'$).

Starting with the state $|\psi\rangle$ Alice and Bob will end up with a state $|\varphi_m\rangle$ with some probability p_m where, at the end of the procedure, they know which $|\varphi_m\rangle$ they have (so they know what m is). The expected entanglement, using the $\log_2 m$ measure is

$$E = \sum_m p_m \log_2 m \tag{3}$$

We will describe a method of manipulating the state by LOCC which can yield the most general transformation of the type $|\psi\rangle \to \{|\varphi_m\rangle; p_m\}$. We will not prove here that this is the most general transformation (for a proof of this the reader is referred to [**5**] (see also [3,6])).

Most of the work is done by Alice. First she introduces an additional quantum system R which she has prepared in the state $|1\rangle_R$. The system R has orthonormal basis states $|n\rangle_R$ where $n = 1$ to N. The initial state is

$$|1\rangle_R|\psi\rangle = \sum_{j=1}^{I} \sqrt{\lambda_j}(|1\rangle_R|j\rangle_A)|j\rangle_B \tag{4}$$

We define the numbers $N_j = N\lambda_j$. We will take $N \to \infty$ so that, at least in this limit, the numbers N_j can be taken to be integers. Alice has the systems R and A under her control. She performs the unitary transformation described by

$$|1\rangle_R|i\rangle_A \to \frac{1}{\sqrt{N_j}}(\sum_{n=1}^{N_j} |n\rangle_R)|i\rangle_A \tag{5}$$

for all j (this transformation is unitary since orthogonal states are mapped onto orthogonal states). After this transformation the state becomes

$$|\Psi_{\text{start}}\rangle = \frac{1}{\sqrt{N}}\sum_j \sum_{n=1}^{N_j} |n\rangle_R|j\rangle_A|j\rangle_B \tag{6}$$

We will call this the *start state.* It is possible to go back to the original state from the start state by reversing the above transformation so there is no change in the entanglement properties at this stage. The start state has the property that each term in it has the same amplitude. It is this property that will make it possible to describe a graphical method for manipulating the entanglement. The terms in the start state are of the form $\frac{1}{\sqrt{N}}|n\rangle_R|j\rangle_A|i\rangle_B$. (Initially $j = i$, however this will not remain true as Alice performs some basic manipulations.) We associate with each such term a rectangle of width 1, height $\frac{1}{N}$, and colour j where this rectangle is placed on a graph with its top right hand corner at position $(\frac{n}{N}, i)$. Each rectangle for all the terms in the start state is plotted on the graph and the resulting graph looks like that shown in figure 1. The colour is not shown, but since initially $j = i$ each column will be of one colour which will be different to the colour of all the other columns. The height of the ith column is λ_i. The area of each rectangle is $\frac{1}{N}$ which is equal to the probability associated with the corresponding term in the start state. The total area is equal to 1.

The basic unitary operation, $U(n, i \leftrightarrow n', i')$, employed by Alice is defined by the transformation equations

$$|n\rangle_R|i\rangle_A \to |n'\rangle_R|i'\rangle_A \tag{7}$$

$$|n'\rangle_R|i'\rangle_A \to |n\rangle_R|i\rangle_A \tag{8}$$

with no change for all other $|n''\rangle_R|i''\rangle_A$. We will call this the swap operation. The effect of this operation is to move elements around on the area diagram. If there are elements at both the (n, i) and (n', i') positions then they will have their positions swapped. If, there is only an element at one of the two positions then it will be moved to the other position while the original position will become vacant. These moves will not effect Bob's part of the state which means that the value of j and hence the colour of each rectangle will not be changed as it is moved to its new position. Hence, by using this swap operation it is possible to redistribute the colour on the graph. Note the following: (i) Alice can move as many rectangles around a once as she likes; (2) As $N \to \infty$ the height of the rectangles tends to zero and hence we can cut along any horizontal line; (3) we can temporarily store pieces in the empty space above the graph. Hence, by local operations Alice can cut and paste the area on the graph in any way she likes as long as the cuts consist of horizontal cuts and vertical cuts between the columns.

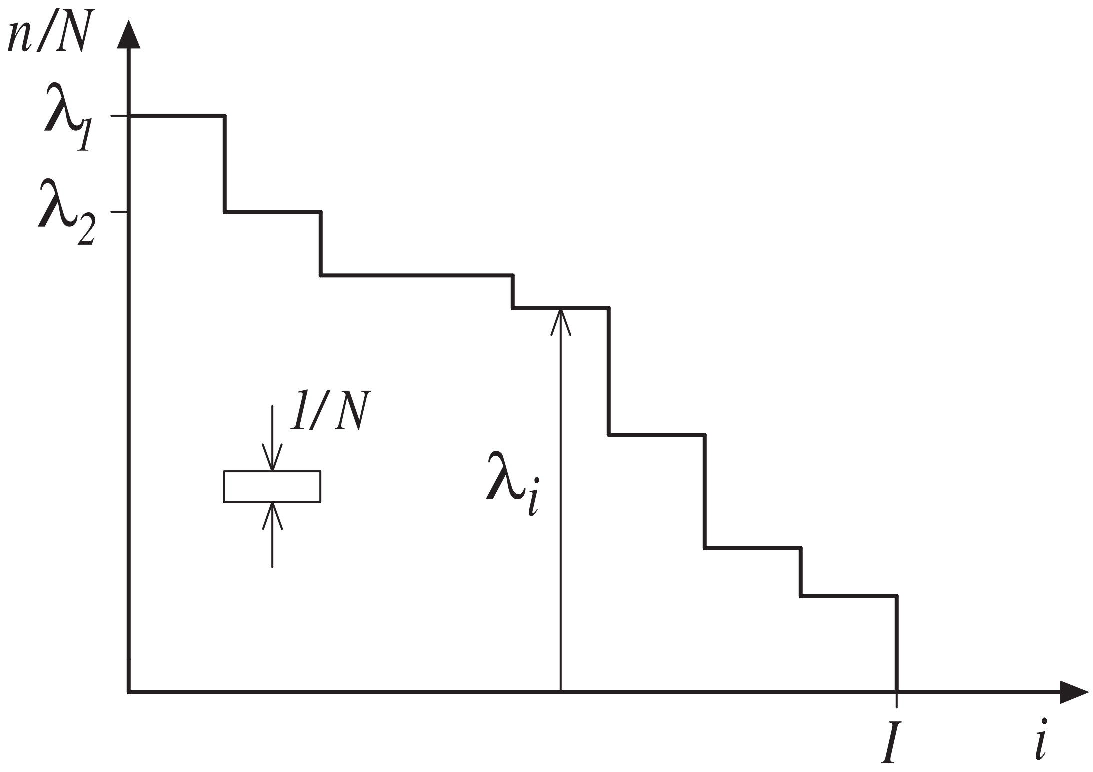

FIGURE 1. The start state can be plotted on a graph. Such a graph is called an area diagram. This area diagram has step structure in which the steps go down towards the right.

Before considering the effect of moving area around, consider what happens if Alice makes a measurement of the value of n by measuring the ancilla R on the basis $|n\rangle_R$. Since we initially have $i = j$ the effect will be to obtain the state (written in unnormalised form)

$$\sum_j |j\rangle_A |j\rangle_B \tag{9}$$

This is an m-state where the value of m is equal to the width of the graph at that value of n. In fact what we have done is pick out a row of rectangles corresponding to a particular value of n. The rectangles in this row all have different values of i (necessarily since they have different horizontal locations on the graph) and, in this case, they also all are of different colour. This second property is essential if the resulting state is to be an m-state. If it were the case that some of the rectangles had the same colour then these terms would not be bi-orthogonal (orthogonal at both Alice's and at Bob's end) and hence the state would not be an m-state.

Now consider moving the area around on the graph. After moving the area around we will make a measurement of n. This will yield a state between Alice and Bob. We want such states to be m-states. Hence we must impose the constraint that after the re-colouring the same colour is not repeated as we go across any given row (for any given n). We can impose a second constraint -

that the new graph also has a step structure in which the steps go down as we go towards the right (i.e. that there are no gaps in the columns and that the height of the columns decreases with i). Hence, the new graph will look like figure 1 except but the columns will have different heights λ'_i (where $\lambda'_i \geq \lambda'_{i+1}$) and the columns can be multicoloured rather than only of one colour. This second constraint (imposing step structure) is not necessary, but it does not lead to any loss of generality and so we can impose it to simplify matters. To see that it does not lead to any loss of generality consider a graph which does not have this property. Nevertheless, a measurement onto n will yield a distribution of m-states (assuming the colours are different across rows). The rows corresponding to these m-states could be contracted by moving all pieces to the leftmost empty position in the row and then these rows could be stacked on top of one another, the widest going at the bottom. In this way we would arrive at a new graph with the same distribution of m-states but now respecting the second assumption. Thus, we want to find the most general way of recolouring the graph in which (a) the new graph has the property that the colours across any row are all different, and (b) the new graph has step structure.

We can see immediately that any net movement of area down is not possible since this will lead to the same colour being in the same row in at least two positions. This means that any successful strategy must involve a net movement of area upwards or not at all. What is surprising is that any new graph having having a net movement of area upwards (or at least not downwards) can be recoloured in accordance with (a) and (b). For a proof of this see [**5**]. Since the new graph must have step structure, a net movement of area upwards (or at least not downwards) is equivalent to a net movement of area to the left. Hence, the new graph must satisfy the algebraic constraints that

$$\sum_{i=p}^{I} \lambda'_i \leq \sum_{i=p}^{I} \lambda_i \tag{10}$$

for all $r = 1$ to I with equality holding when $r = 1$. These constraints follow because the area to the right of any vertical line drawn between two columns must have less area to its left afterwards

After performing swap transformations corresponding to an appropriate recolouring, Alice will make a measurement of n and communicate the result to Bob. For this value of n Alice and Bob can read from the final graph what state they have. This state will always be an m-state. If they want to put their state into the standard form in (2) then Bob can perform an appropriate unitary transformation. For example, they might end up with the 2-state $|1\rangle_A|3\rangle_B + |2\rangle_A|4\rangle_B$. Bob can perform the unitary transformation defined by $|3\rangle_B \leftrightarrow |1\rangle_B$ and $|4\rangle_B \leftrightarrow |2\rangle_B$ which will put the state in the standard form $|1\rangle_A|1\rangle_B + |2\rangle_A|2\rangle_B$.

The probability of obtaining some particular value of m for the m-state is equal to the area of rectangle obtained by extending back horizontal lines from the top and bottom of the step having width m, i.e. $p_m = (\lambda'_m - \lambda'_{m+1})m$. Hence, the expected entanglement using the $\log_2 m$ measure is

$$E = \sum_i (\lambda'_i - \lambda'_{i+1}) i \log_2 i \tag{11}$$

In fact any net movement of area to the left will decrease E (this is not obvious but follows from the convex nature of the log function) and hence the maximum expected entanglement is

$$E_{\max} = \sum_i (\lambda_i - \lambda_{i+1}) i \log_2 i \tag{12}$$

corresponding to the original graph. For a full proof that this is the maximum see [**5**] (see also [**6**]).

The main advantage of the approach described above is that the transformations required can be pictured as graph transformations. This method can be applied to other problems concerning the manipulation of the entanglement of a single copy of a pure state. For example, it can be used to perform the transformation, by LOCC, of one entangled state to another in accordance with Nielsen's theorem. This method does not, however, generalise easily multipartite pure states. The principle reason for this is that there is no Schmidt decomposition for such states in general. This leaves open multipartite entangled states as an area where exciting developments are likely to happen in the near future.

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