Five Dimensional Knot Theory

S. J. Lomonaco, Jr.*
University of Maryland Baltimore County
Baltimore, MD 21250
Email: Lomonaco@UMBC>EDU
WebPage: http://www.cs.umbc.edu/~lomonaco

Published in AMS Contemporary Mathematics, Vol. 20, (1983), pp. 249 - 270.

Abstract. This paper outlines in an informal and intuitive fashion a method for computing the second homotopy group $\pi_2 X$ (as a left $\mathbb{Z}\pi_1 X$-module) of the complement $X = S^5 - kM^3$ of a smooth 3-knot $\left(S^5\ ,\ kM^3\right)$. By a smooth 3-knot $\left(S^5\ ,\ kM^3\right)$ is meant a smooth imbedding of a closed 3-manifold M^3 into the 5-sphere S^5. An example of the calculation is given.

The methods of this paper may also be used to compute the Rohlin invariant of 3-manifolds from their imbeddings in S^5.

1. Introduction

The purpose of this paper is to outline in an informal and intuitive fashion a solution to the following problem.

PROBLEM 1: How does one compute the homotopy group $\pi_2 X$ (as a left $\mathbb{Z}\pi_1 X$-module) of the complement $X = S^5 - kM^3$ of a 5-D knot [i.e., a 3-knot] $(S^5\ ,\ kM^3)$?

A more detailed and rigorous exposition of the results contained herein will appear in [7].

This paper follows a "divide and conquer" philosophy by separating this problem into two essential components, namely:

PROBLEM 1a: How does a 3-D creature visualize a 5-D knot, i.e., a 3-knot $(S^5\ ,\ kM^3)$?

PROBLEM 1b: How does such a creature compute the homotopy group $\pi_2 X$ (as a left $\mathbb{Z}\pi_1 X$-module) of the complement $X = S^5 - kM^3$ from the above visualization?

*Supported in part by the L-O-O-P Fund.

2. PRELIMINARY DEFINITIONS, CONVENTIONS, AND OBSERVATIONS

Before proceeding further, we need to give some preliminary definitions, conventions, and observations.

Definition 1. *An* **n-knot**, *also called an* **(n + 2)-D knot**, *denoted by* $(S^{n+2}\ ,\ kM^n)$ *is a smooth imbedding* $k : M^n \longrightarrow S^{n+2}$ *of a smooth closed* n*-manifold* M^n *into the* $(n+2)$*-sphere* S^{n+2}. *Unless explicitly stated otherwise,* X *will always denote its* ***complement*** $S^{n+2} - kM^n$.

Remark 1 [Convention]**.** *To avoid cluttering this article with needless technical detail, we assume (unless explicitly stated otherwise) that the manifold* M^n *is both orientable and connected. The reader may easily extend the results of this paper to non-orientable and/or non-connected manifolds* M^n.

There is a beautiful interplay between n-knots and n-manifolds that can lead to a better understanding of both research areas. For, example, by Alexander duality, we have:

$$\overline{H}_q S \cong \overline{H}_{n+1-q} M^n,$$

where $\overline{H}_*$ denotes the reduced homology and X (as per our convention) denotes the complement $X = S^5 - kM^3$. Thus,

- **(a)** $H_1 \cong \mathbb{Z}$
- **(b)** X is a homology 1-sphere if and only if M^n is a homology n-sphere.

Moreover, this interrelationship can also be used, for example, to compute the Rohlin invariant of 3-manifolds from their imbeddings in the 5-sphere S^5.

Remark 2. *For the complement* X *of 2-knots, we have:*

- **(a)** X is a homology 1-sphere if and only if M^2 is a 2-sphere,
- **(b)** The second betti number $\beta_2(X)$ of the complement X is equal to twice the genus $g(M^n)$ of M^2, i.e.,

$$\beta_2(X) = 2g(M^n)\ .$$

3. A MOMENTARY DIGRESSION: 4-D KNOT THEORY

Before solving Problem 1a of Section I, we first need to gain some geometric intuition by considering the same problem in dimension 4 instead of 5, i.e.,

Problem 2a: How does a three dimensional creature visualize a knotted closed surface (i.e., closed 2-manifold) M^2 lying in 4-space?

Such a three dimensional creature could think of 4-space $\mathbb{R}^4$ as spanned by a parameterized family of 3 dimensional hyperplanes $\mathbb{R}^3_t = \mathbb{R}^3 \times t$, i.e.,

$$\mathbb{R}^4 = \bigcup_{t\in\mathbb{R}} \mathbb{R}^3_t \ ,$$

where the parameter t, called **time**, varies over the real numbers $\mathbb{R}$. He (or she) could then visualize an object lying in 4-space by letting the 3 dimemsional hyperplane in which he (or she) lived move through time t to view the 3 dimensional cross sections of the object as they appear to be moving panoramically by as time t passes.

Since we will be working in the 4-sphere S^4, we will add the point at infinity ∞ to $\mathbb{R}^4$ and to each hyperplane $\mathbb{R}^3_t$ (i.e., one point compactify) to obtain:

$$S^4 = \mathbb{R}^4 \cup \infty = \bigcup_{t\in\mathbb{R}^1} S^3_t \ ,$$

where S^3_t is the 3-sphere $\mathbb{R}^3_t \cup \infty$, and where $S^3_t \cap S^3_{t'} = \infty$ when $t \neq t'$.

Each one pont compactified hyperplane S^3_t will be called a **snapshot** or **frame**. The entire series of frames will be called a **motion picture** or **movie**!

A movie of a trivially imbedded 2-sphere S^2 in the 4-sphere S^4 can be found in Fig. 1. One of a knotted 2-sphere S^2 in S^2 can be found in Fig. 2. In Figs. 1 and 2, the "bottom valley" points e_-, e'_- and the "peak" points e_+, e'_+ are respectively the Morse index 0 and 2 singularities. Collectively, e_-, e'_-, e_+, e'_+ are called the **elliptic points** of the imbedded surface. The **saddle points** or **hyperbolic points** h, h' are the Morse index 1 singularities. Finally, if $X = S^4 - kM^2$ denotes the complement of a 4-D knot $(S^4,\ kM^2)$ (as per our convention), then $X_t = X \cap S^3_t$ will denote the 3 dimensional cross section or **frame at time** t.

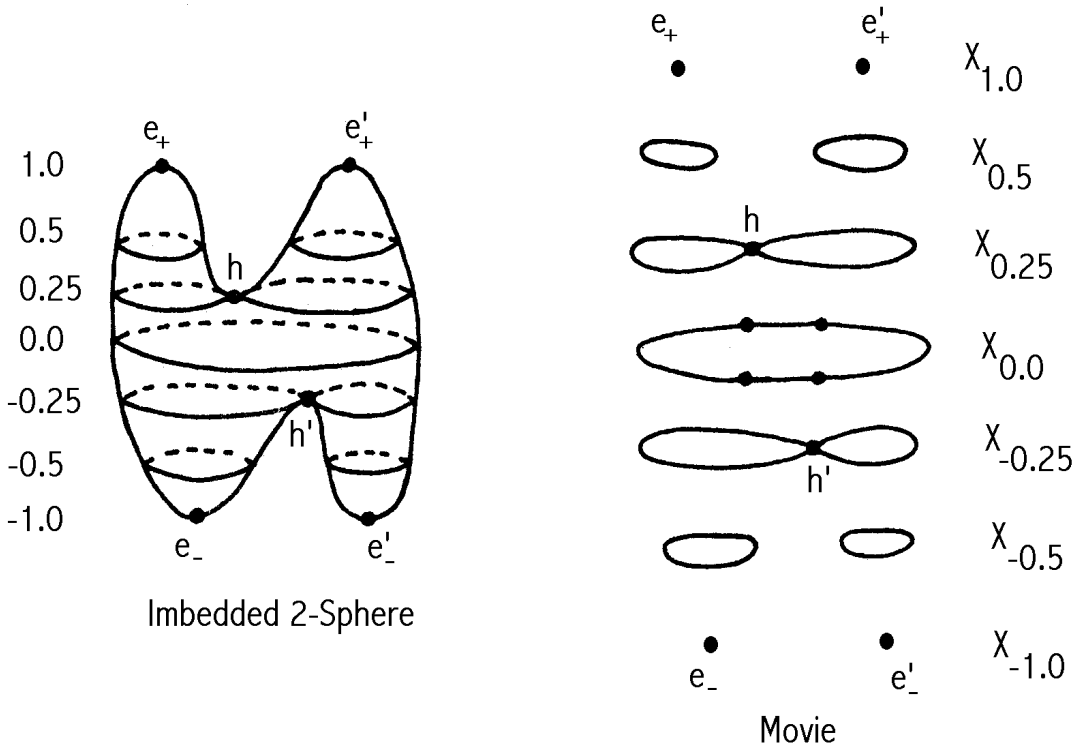


Figure 1. A Movie of a Trivially Embedded 2-Sphere in S^4.

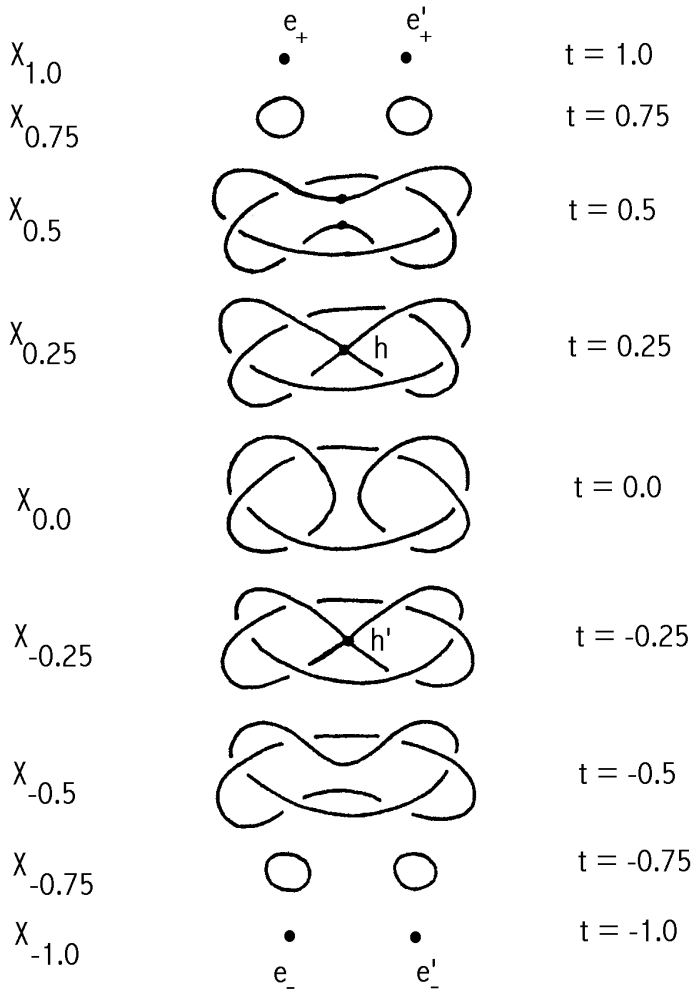


Figure 2. A Movie of a Knotted 2-Sphere in S^4.

The above method for visualizing 4-D knots was devised by R. H. Fox in [**?**]. We now show how this method of representation can be greatly simplified by first **normalizing** each movie and then by pointing out that, surprisingly enough, essentially only one single 3-dimensional cross section, the **key frame**, in the normalized movie is needed to represent the knot!

We normalize each movie by pushing all hyperbolic points into the same 3-dimensional hyperplane, i.e., the hyperplane at time $t = 0$, called the **key frame**, and pushing all elliptic points of type 0 and 2 into the frames at times $t = -1$ and $+1$ respectively. (See [6] for a more formal and rigorous definition.) Thus, the movies in Figs. 1 and 2 respectively become after normalization the movies given in Figs. 3 and 4.

A quick glance at the normalized movies found in Figs. 3 and 4 will convince the reader that all such normailized movies are highly redundant representations of 2-knots. In fact, each entire normalized movie can be completely reconstructed from the key frame $X_0 = X \cap S^3_0$ (i.e., the frame containing all the hyperbolic points) together with a set of labels (one for each hyperbolic point) indicating how each hyperbolic point opens up above, i.e., for $t > 0$.

The result is called the **key frame representation** of the 4-D knot. Thus, using the labeling scheme defined in Fig. 5, we have in Figs. 6 and 7 the key frame representation of the 4-D knots found in Figs. 1, 3 and 2, 4 respectively. Key frame representations of other 2-knots can be found in [6, Figs. 4, 5, 6, 7, 9].

In summary, we have found a very simple and concrete means of representing 4-D knots, the key frame representation. In fact, this representation reduces the study of 4 dimensional knots to the study of labeled knotted graphs in S^3.

Remark 3. *The key frame reprsentation (called the 2-knot diagram in [6]) was devised in [6] to compute $\pi_1 X$, $\pi_2 X$ as a left $\mathbb{Z}\pi_1 X$-module, and the k-invariant kX lying in $H^3(\pi_1 X;\ \pi_2 X)$ of the complement X of 2-knots $(S^4,\ kM^2)$. The results of these calculations are given in Fig. 7 and in [6, Figs. 5, 6, 7, 9].*

Remark 4. *The key frame representation of 2-knots is analogous in many ways to Heegard diagrams of 3-manifolds. (See [4].)*

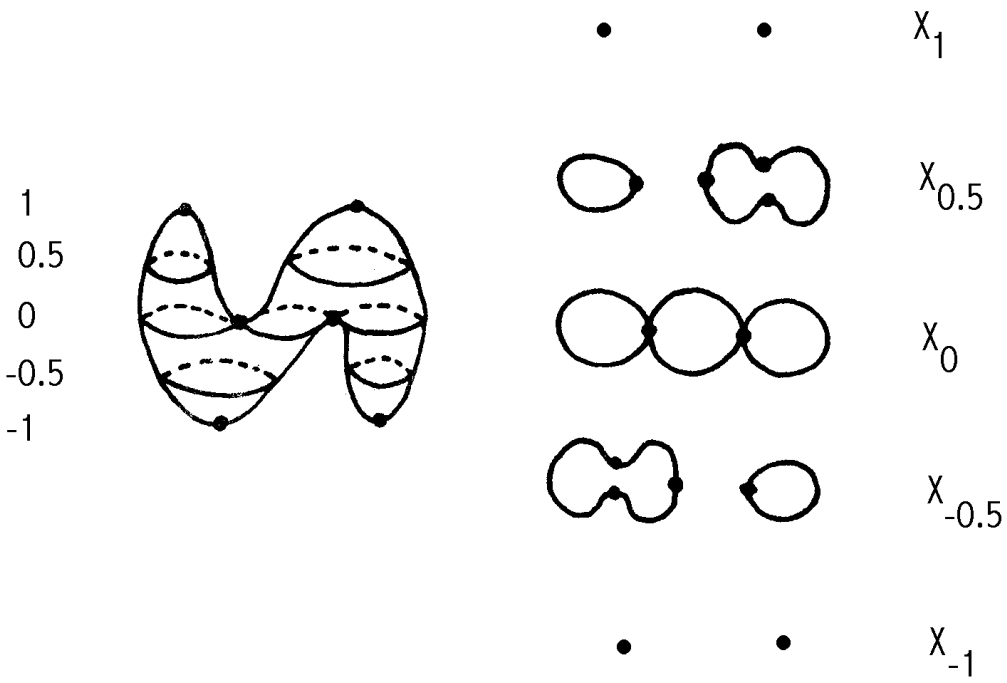


Figure 3. A normalized movie of the example found in Figure 1. All the hyperbolic points have been pushed into the frame X_0 (called the KEY FRAME).

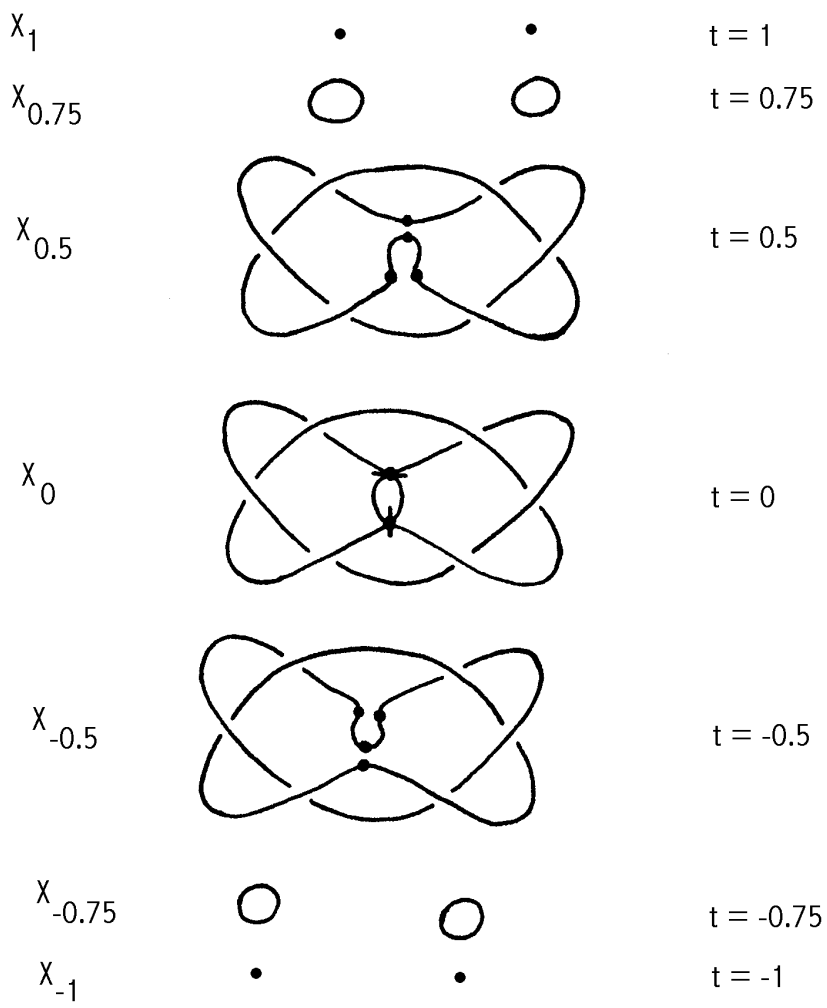


Figure 4. A normalized movie of the example found in Figure 2. All hyperbolic points have been pushed into yhe frame X_0 (called the key frame).

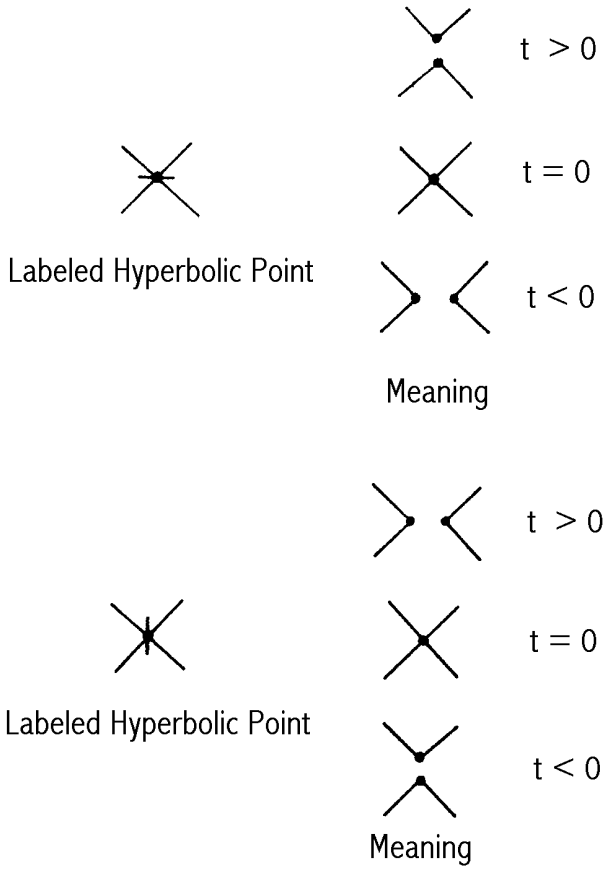


Figure 5. Labeling scheme for hyperbolic points

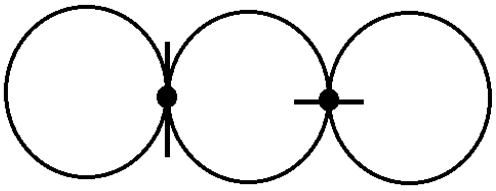
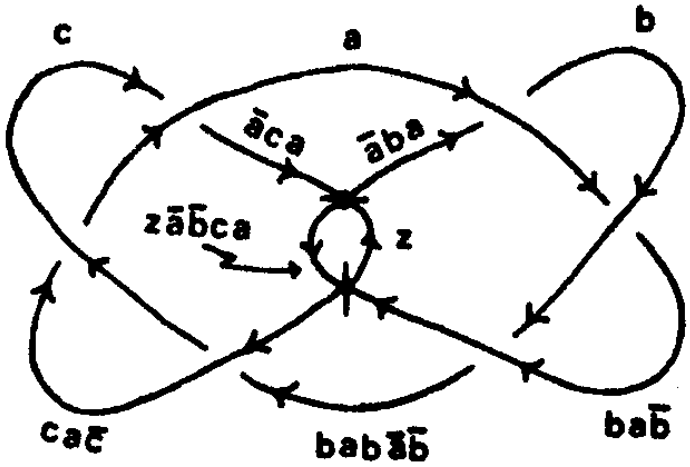


Figure 6. Key frame representation of example found in Figures 1 and 3.



$$\begin{cases} \pi_1 K = (\ a, b : abab^{-1}a^{-1}b^{-1} = 1\ ) \\ \pi_2 K = (\ \partial U_{\square} :\ (1\text{-}a\text{+}ba)\,\partial U_{\square} = 0\ ) \\ H^3(\ \pi_1 K ; \pi_2 K) = 0 \quad \text{and} \quad \underline{\underline{k}} K = 0 \end{cases}$$

Figure 7. Key frame representation of example found in Figures 2 and 4.

4. Visualizing 5-D Knots

Armed with the geometric intuition of Section III, we can now solve problem 1a, i.e.,

Problem 1a: How does a 3-D creature visualize a 5-D knot, i.e., a 3-knot $(S^5,\ kM^3)$?

Imitating Section III one dimension up, we can think of the 5-sphere $S^5 = \mathbb{R}^5 \cup \infty$ as spanned by a parameterized family of 4-spheres $S^4_s = (\mathbb{R}^4 \times s) \cup \infty$, i.e.,

$$s^5 = \bigcup_{s \in \mathbb{R}} S^4_t$$

where the parameter s, called **hypertime**, varies over the real numbers $\mathbb{R}$. Thus, if we were 4 dimensional creatures, we could visualize an object in the 5-sphere

S^5 by letting the 4-sphere in which we lived move through positive hypertime s to view the 4 dimensional cross sections of the object as they passed by. Thus, if $X = S^5 - kM^3$ denotes the complement of a 3-knot $(S^5,\ kM^3)$ [as per our convention], then $X_s = X \cap S^4_s$ will denote the 4 dimensional cross section of the complement at hypertime s.

However, we are mere 3 dimensional creatures. So we must think of each 4-D section X_s as a movie of 3-D frames

$$X_{st} = X_s \cap S^3_{st} \ ,$$

where $S^3_{st} = (\mathbb{R}^3 \times s \times t) \cup \infty$ and where

$$S^4_{st} = \bigcup_{t \in \mathbb{R}} S^3_{st} \text{ and } X_s = \bigcup_{t \in \mathbb{R}} X_{st} \ .$$

Hence, we are forced by our very nature to think of 5-D knots in terms of a family X_s of movies, i.e., in terms of a **movie of movies**!

A movie of movies of a 3-knot $(S^5,\ kS^3)$ is given in Fig. 8. Each vertical strip X_s of this figure is a movie of a possibly singular surface in S^4 and represents the 4-dimensional cross section of the 3-knot complement at hypertime s. each square X_{st} is a **frame** [at (hypertime, time) $= (s, t)$] representing a 3 dimensional cross section at time t of a 4 dimensional cross section at hypertime s. By convention, time t moves from bottom to top and hypertime s drom left to right.

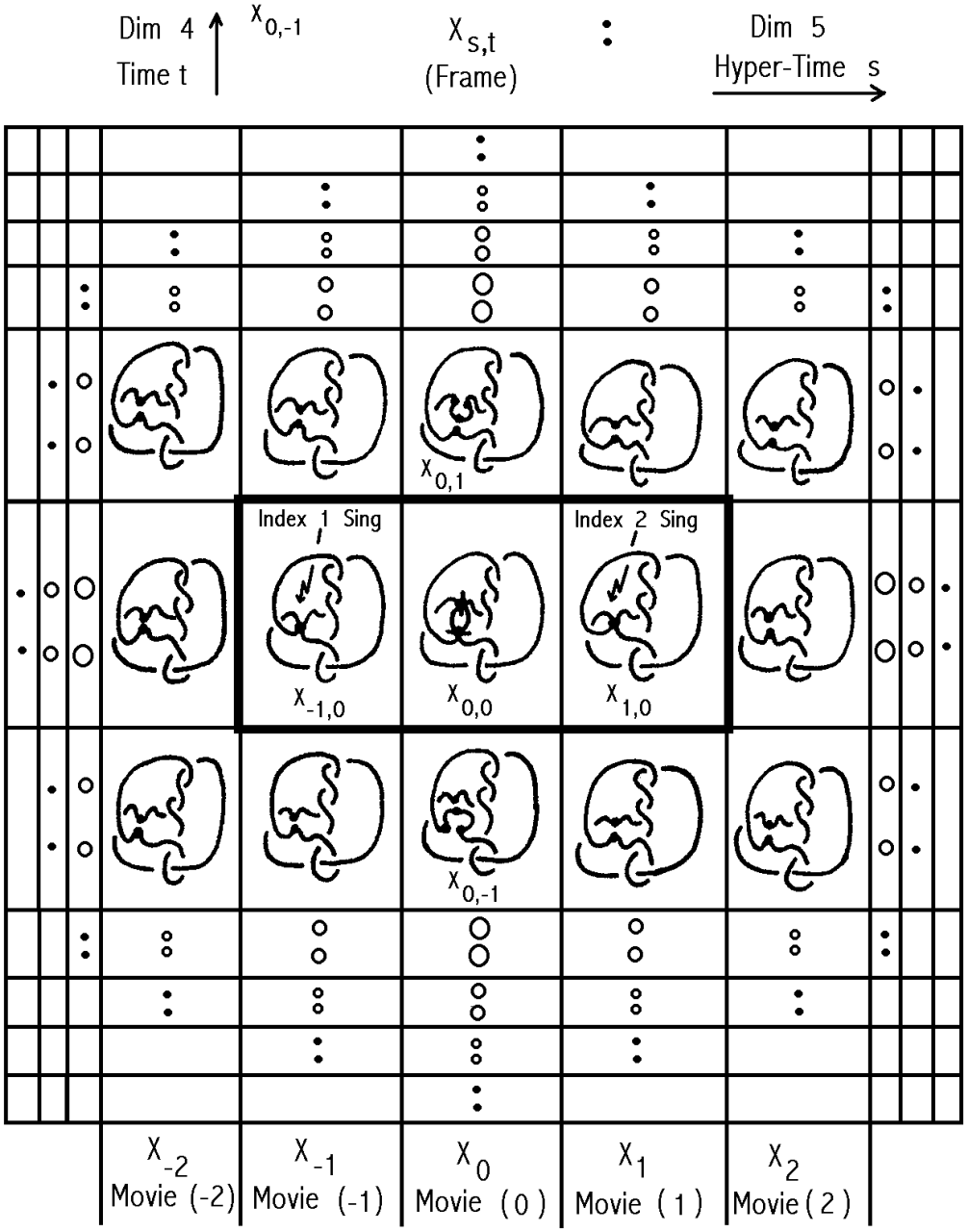
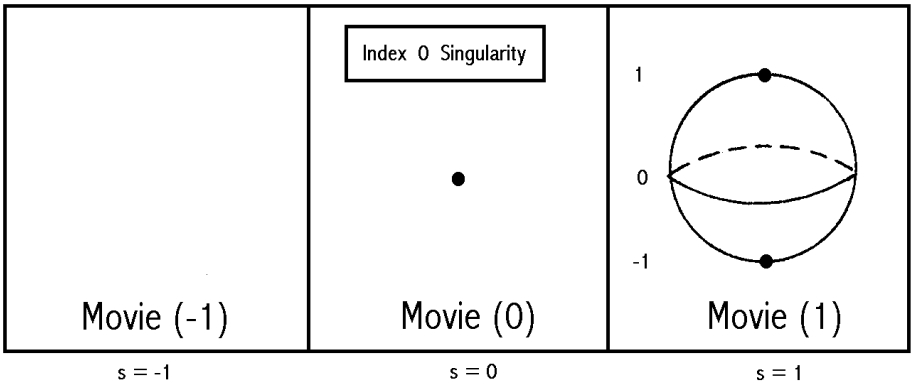
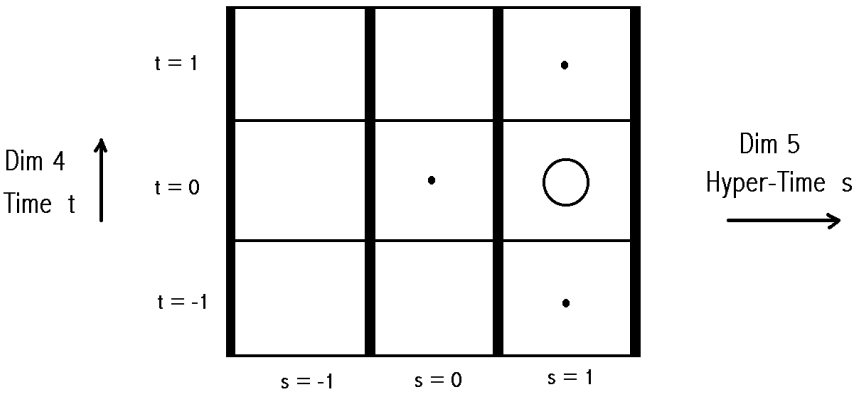


Figure 8. Movie of movies of 3-knot (S^5, kS^3).

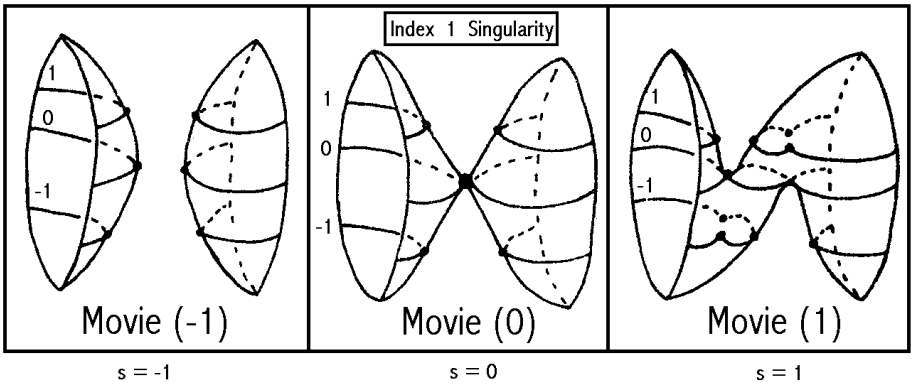


Part A. Each square above represents a 3 dimensional snapshot of the corresponding local movie (i.e., vertical strip) in Part B below.

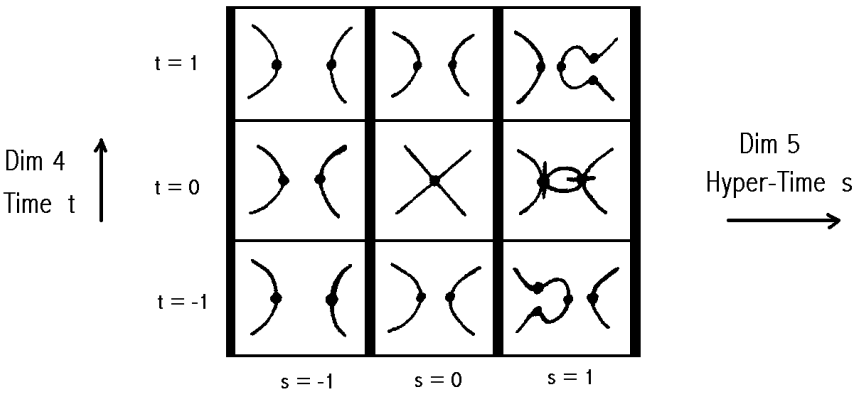


Part B. A local movie of movies of an index 0 singularity.

Figure 9. Index 0 Morse singularity.

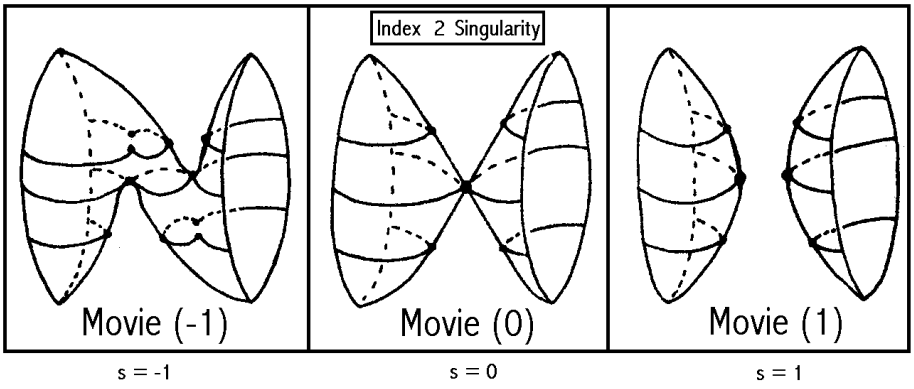


Part A. Each square above represents a 3 dimensional snapshot of the corresponding local movie (i.e., vertical strip) in Part B below.

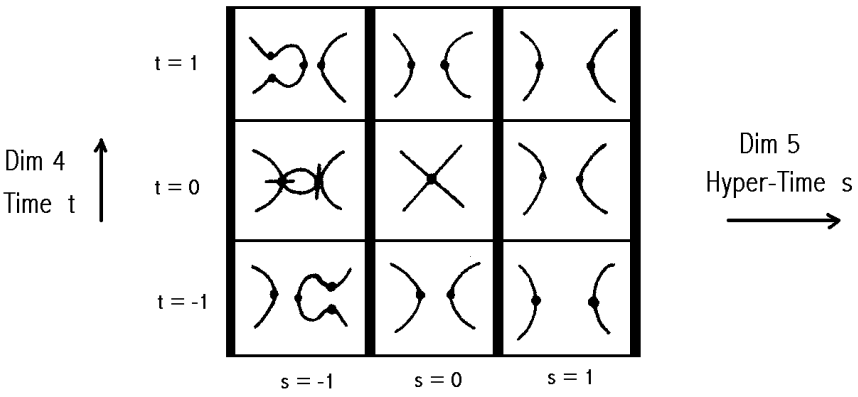


Part B. A local movie of movies of an index 1 singularity.

Figure 10. Index 1 Morse singularity.

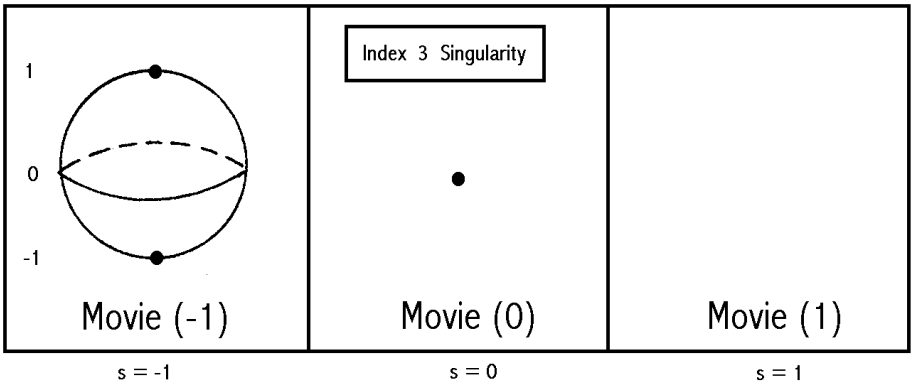


Part A. Each square above represents a 3 dimensional snapshot of the corresponding local movie (i.e., vertical strip) in Part B below.

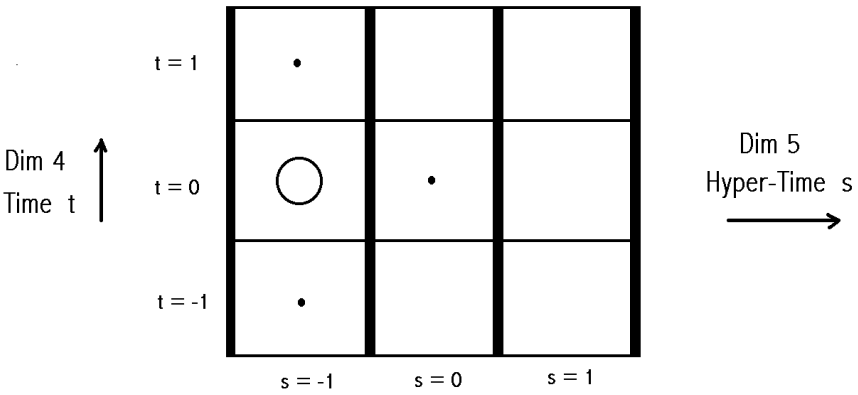


Part B. A local movie of movies of an index 2 singularity.

Figure 11. Index 2 Morse singularity.



Part A. Each square above represents a 3 dimensional snapshot of the corresponding local movie (i.e., vertical strip) in Part B below.



Part B. A local movie of movies of an index 3 singularity.

Figure 12. Index 3 Morse singularity.

The movie of movies representation of a 3-knot $(S^5,\ kM^3)$ may at first glance look hopelessly complex. Bus, as in Section III, it can be vastly simplified by normalizing. For we shall see that, after normalization, essentially **only three** key frames are needed to represent a 5-D knot!

We first note that there are four types of Morse singularities that can arise, i.e., Morse sigularities of index o, 1, 2, and 3. An example of each of these singularities can be found within the example given in Fig. 8. A local movie of movies of index 0, 1, 2, 3 singularities are given respectively in part b of Figs. 9 through 12. The squares in each part a of Figs. 9 through 12 represent 3 dimensional snapshots of the corresponding local movies (i.e., the corresponding vertical strips) in each part b.

We can normalize a movie of movies by pushing all index 0, 1, 2, 3 singularities into movies X_{-5}, X_{-1}, X_1, X_{-5}, respectively, then by pushing the respective singularities into frames $X_{(-5,0)}$, $X_{(-1,0)}$, $X_{(1,0)}$, $X_{(5,0)}$. Figure 8 gives a normalized movie of movies.

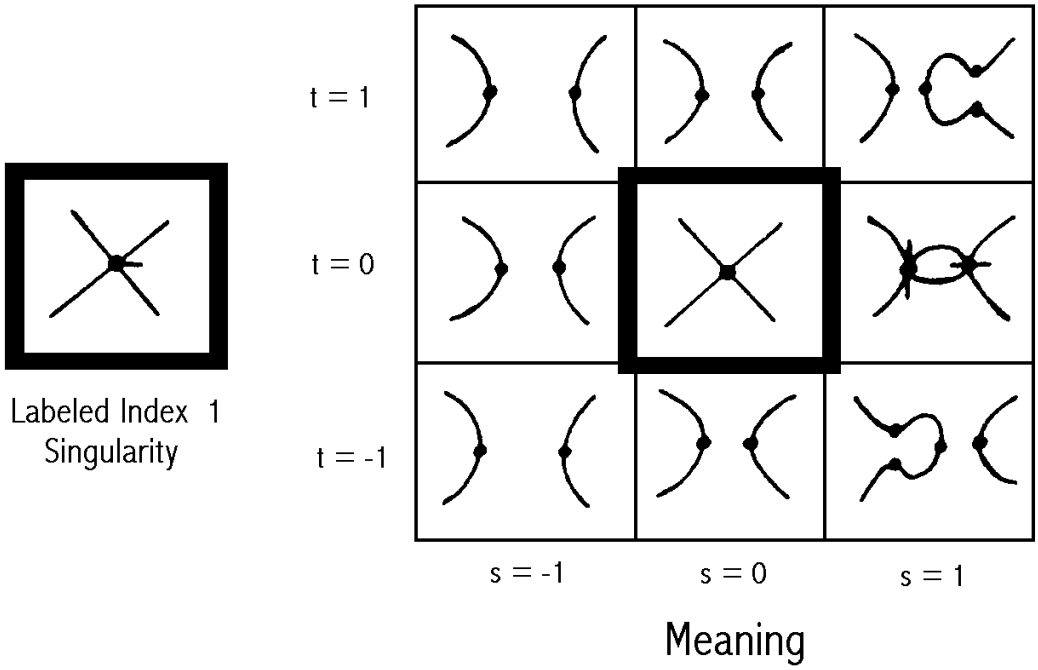


Figure 13. Labeling scheme for index 1 singularity.

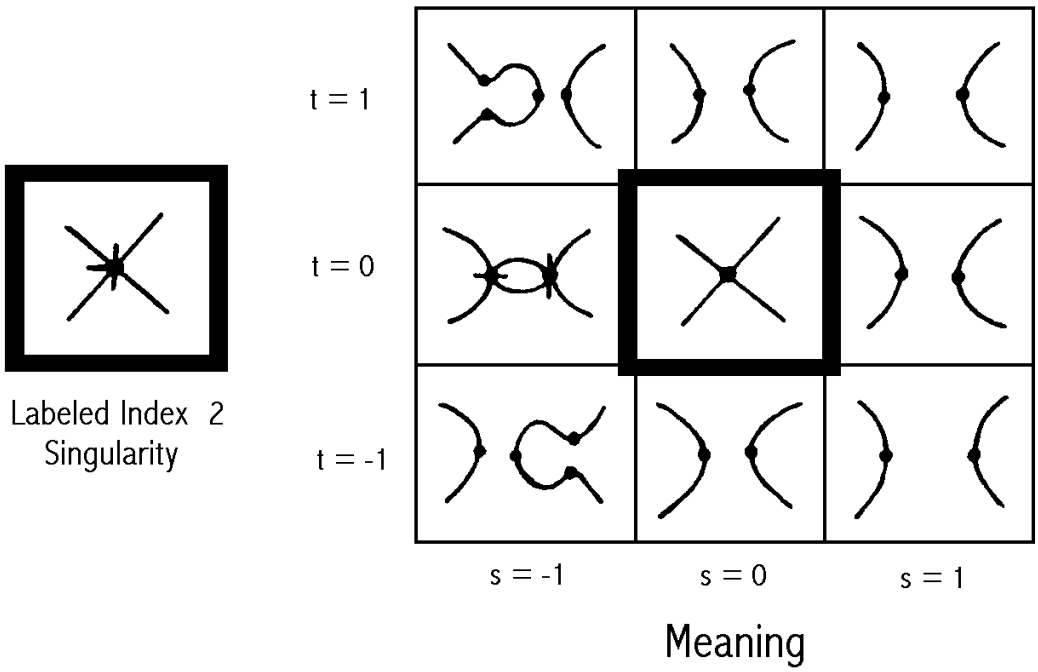


Figure 14. Labeling scheme for index 2 singularity.

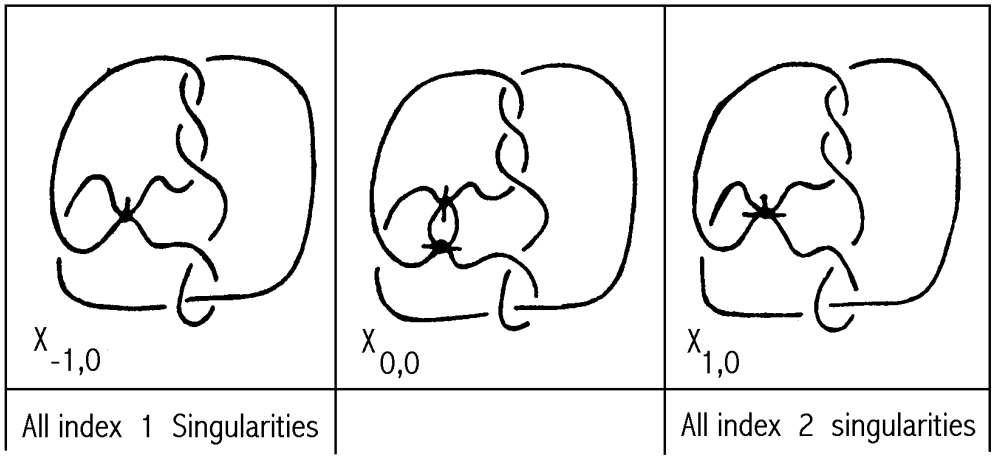


Figure 15. Key frame representation of the 3-knot (S^5, kS^3) given in Figure 8.

As in Section III, a quick glance at the normalized movie of movies found in Fig. 8 will convince the reader that all such normalized movie of movies are highly redundant representations of 3-knots. In fact, each entire normalized movie of movies

can be respectively reconstructed from the three frames $X_{-1,0}$, $X_{0,0}$, $X_{1,0}$, called the **key frames**, together with a set of labels (one for each index 1 and one for each index 2 singularity) indicating how each singularity opens as time and hypertime pass.

The result is called a **key frame representation** of a 5-D knot. Thus, using the labeling scheme defined in Figs. 13 and 14, we have in Fig. 15 the key frame representation of the 5-D knot found in Fig. 8.

In summary, we have solved problem 1a by devising a concrete means, i.e., the key frame representation, for a 3-D creature to visualize objects lying in 5 dimensional space.

Remark 5. *Actually all index 1 and index 2 singularities in Fig. 8 can be pushed into frame $X_{0,0}$. The resulting movie of movies is shown in Fig. 16. Hence, for this example, only one labeled key frame is needed to represent the entire 3-knot. However, I do not believe that this single frame representation is possible in general.*

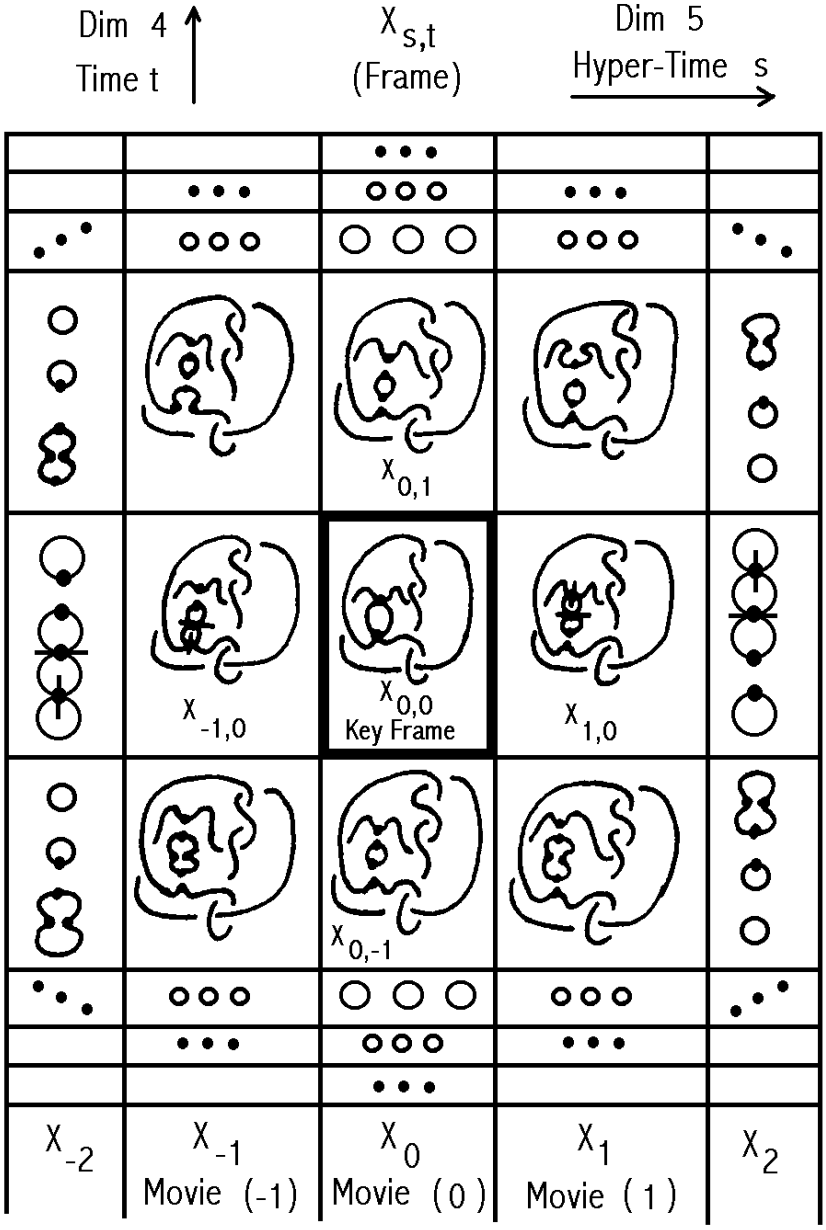


Figure 16. A non-standard movie of movies of 3-knot in Figures 8 and 15.

5. A Strategy for Computing π_* of 5-D Knots

In the next section, explicit calculations for π_2 and π_3 of the complement of the 5-D knot found in Figs. 8 and 15 will be given. This section is devoted to outlining

a general strategy for computing π_2 and in some cases π_3 of the complement X of an arbitrary smooth 5-D knot $(S^5,\ kM^3)$ from the key frame representation method devised in the previous section. We assume that the reader has some understanding of the homology of group systems. (See [5], [7], [8].)

Step 1: Divide the complement X into a finite collection $\underline{\underline{\mathbf{X}}}=\{X_\alpha\}$ of connected aspherical building blocks X_α such that

a) $X = \cup\underline{\underline{\mathbf{X}}}$.

b) For every subcollection $\underline{\underline{\mathbf{X}}}'$ of $\underline{\underline{\mathbf{X}}}$, the intersection $\cap\underline{\underline{\mathbf{X}}}'$ is connected.

c) For every subcollection $\underline{\underline{\mathbf{X}}}'$ of $\underline{\underline{\mathbf{X}}}$ there exists a subcollection $\underline{\underline{\mathbf{X}}}''$ of $\underline{\underline{\mathbf{X}}}$ such that

$$\cap\underline{\underline{\mathbf{X}}}' = \cup\underline{\underline{\mathbf{X}}}''.$$

Step 2: Construct frm $\underline{\underline{\mathbf{X}}}$ the **associated group system** $\pi_1\underline{\underline{\mathbf{X}}}$, i.e., the group system consisting of all the fundamental groups $\pi_1 X_\alpha$ together with all the morphisms $\pi_1 X_\alpha \longrightarrow \pi_1 X_\beta$ induced by inclusion $X_\alpha \subset X_\beta$.

Step 3: By the Van Kampen theorem [1], $\pi_1 X$ is the push out of $\pi_1\underline{\underline{\mathbf{X}}}$, i.e.,

$$\pi_1 X = Push\ \pi_1\underline{\underline{\mathbf{X}}}\ .$$

Step 4: By theorem 4 of [5] (theorem 7.1 of [6]) coupled with the Hurewicz theorem,

$$\pi_2 X = H_2\left(\pi_1\underline{\underline{\mathbf{X}}};\ \mathbb{Z}\pi_1 X\right)$$

as a left $\mathbb{Z}\pi_1 X$-module.

Step 5: Finally, if $\pi_2 X = 0$, then by the same theorems quoted in step 4,

$$\pi_3 X = H_3\left(\pi_1\underline{\underline{\mathbf{X}}};\ \mathbb{Z}\pi_1 X\right)$$

as a left $\mathbb{Z}\pi_1 X$-module.

6. EXPLICIT CALCULATIONS FOR $\pi_* X$

The homotopy groups $\pi_2 X$ and $\pi_3 X$ (as left $\mathbb{Z}\pi_1 X$-modules) of the complement X of the 3-knot found in Figs. 8 and 15 are explicitly computed below by using the computational schema of the previous section.

We will implement step 1 by constructing an aspherical decomposition of X that is peculiar to our particular example. The reader, if he (or she) wishes, may use the

more general aspherical decomposition (based on the 3 frame representation) giiven in the appendix. This will lead to a lengthier computation that reveals nothing essentially different from the shorter computation below.

As mentioned in the remark at the end of section V, the index 1 and 2 singularities of the example given in Figs. 8 and 15 can all be pushed into the single frame X_{00} to form the movie of movies given in Fig. 16 and the single frame representation given in Fig. 17. We caution yhat this probably is not always possible for all 3-knots, i.e., not every 3-knot has a single frame representation.

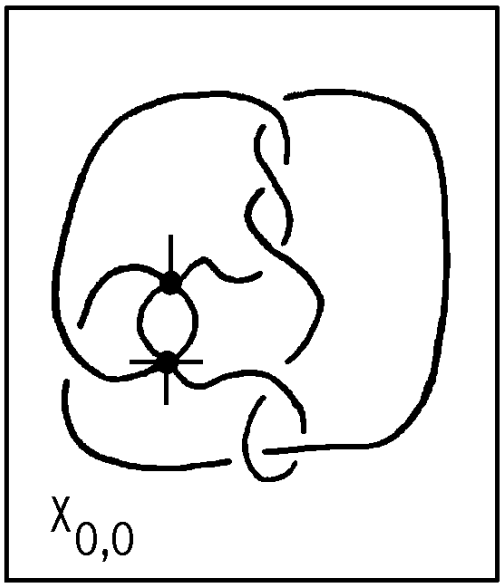


Figure 17. a non-standard single frame representation of the 3-knot given in Figures 8, 15, and 16.

We implement step 1 of the computation schema using the representation given in Figs. 16 and 17.

Step 1: Slice X along X_0 into two pieces X_L and X_R ("L" and "R" for "left" and "right" respectively) such that

$$\begin{cases} X &= X_L \cup X_R \\ X_0 &= X_L \cap X_R \end{cases}.$$

Both X_L and X_R are aspherical because they collapse to the wedge of 1-spheres. Since, unfortunately, X_0 is not aspherical, we continue the subdivision.

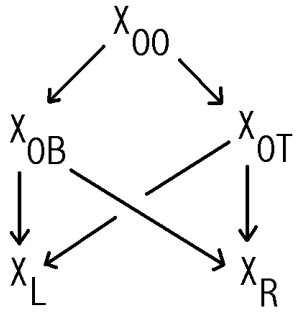
Cut X_0 along X_{00} into two pieces X_{0B} and X_{0T} ("B" and "T" for "bottom" and "top" respectively) so that

$$\begin{cases} X_0 &= X_{0B} \cup X_{0T} \\ X_{00} &= X_{0B} \cap X_{0T} \end{cases} .$$

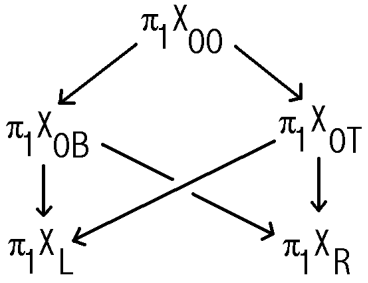
X_{0B} and X_{0T} are aspherical since they collapse to the wedge of 1-spheres; and fortunately, X_{00} is also aspherical since it is the complement of a connected graph in S^3. Thus,

$$\underline{\underline{\mathbf{X}}} = \{X_L,\ X_R,\ X_{0B},\ X_{00},\ X_{0T}\}$$

is an aspherical structure on X. The inclusin maps among the elements of $\underline{\underline{\mathbf{X}}}$ are indicated by the arrows in the diagram below.



Step 2: The group system $\pi_1\underline{\underline{\mathbf{X}}}$ induced by $\underline{\underline{\mathbf{X}}}$ is:



Step 3: $\pi_1 X \ = \ Push\ \pi_1\underline{\underline{\mathbf{X}}}$

$$\therefore \pi_1 X \ = \ (\ a,\ x,\ y,\ z,\ u,\ v\ :\ r_{L1},\ r_{L2},\ r_{0B},\ r_{00},\ r_{0T},\ r_{R1},\ r_{R2}\ )$$

$$= \ (\ a,\ x\ :\ ax = xa^2)\ ,$$

where

$$\begin{cases} r_{L1} &= y\overline{u} \\ r_{L2} &= x\overline{u} \\ r_{0B} &= y\overline{u} \\ r_{00} &= a(y\overline{z})x(z\overline{y})\overline{x}\ \overline{a}(y\overline{z}) \end{cases} \qquad \begin{cases} r_{0T} &= y\overline{z} \\ r_{R1} &= ax\overline{a}^2\overline{v} \\ r_{R2} &= z\overline{v} \end{cases}$$

where $\overline{u}$, $\overline{z}$, etc. denote respectively u^{-1}, z^{-1}, etc. (See Fig. 18.)

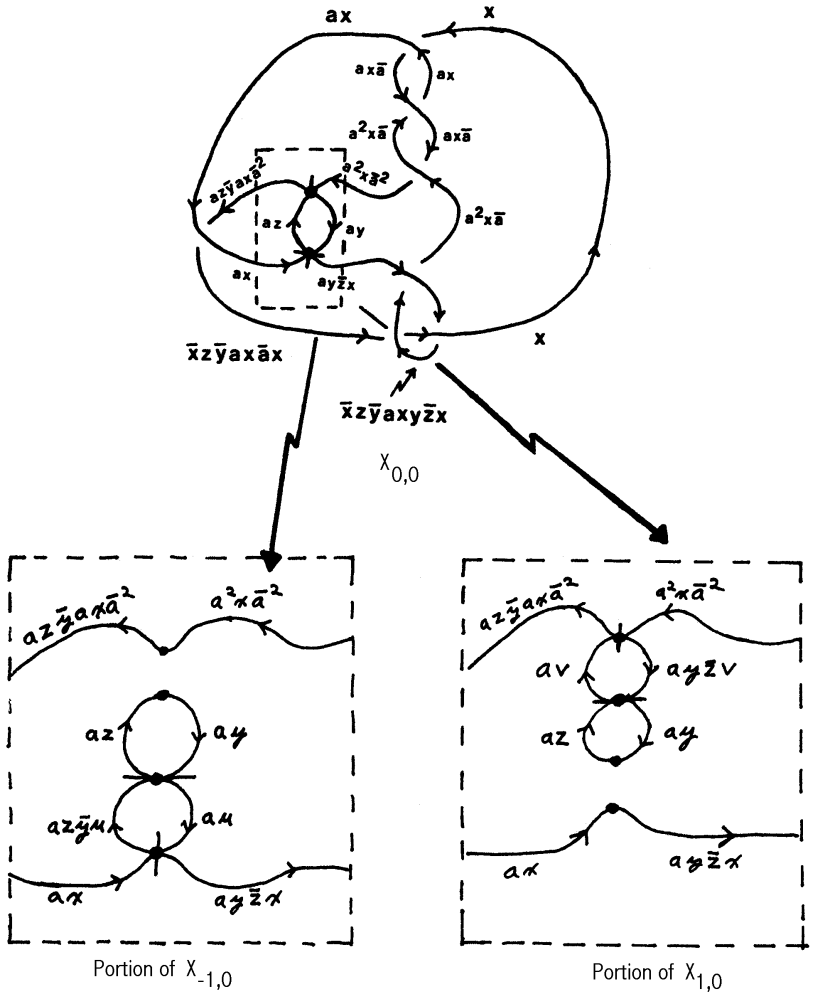


Figure 18. Wirtinger generators for a presentation of the fundamental group $\pi_1 X$ of the complement of the 3-knot given in Figure 17.

Step 4:

$$\begin{aligned}\pi_2 X &= H_2\left(\pi_1 \underline{\underline{\mathbf{X}}};\ \mathbb{Z}\pi_1 X\right)\\ &= H_2 C\pi_1 \underline{\underline{\mathbf{X}}}\end{aligned}$$

where the left $\mathbb{Z}\pi_1 X$ chain complex $C\pi_1 \underline{\underline{\mathbf{X}}}$ is computed from Fig. 18 to be:

$$\ldots \to 0 \to C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where

$\mathbb{Z}$ is the infinite cyclic group as a trivial left $\mathbb{Z}\pi_1 X$-module

C_0 is the free left $\mathbb{Z}\pi_1 X$-module generated by the symbol P

C_1 is the free left $\mathbb{Z}\pi_1 X$-module on the symbols $A,\ X,\ Y,\ Z,\ U,\ V$

C_2 is the free left $\mathbb{Z}\pi_1 X$-module on the symbols $R_{L1},\ R_{L2}, R_{0B}, R_{00}, R_{0T}, R_{R1}, R_{R2}$

C_3 is the free left $\mathbb{Z}\pi_1 X$-module on the symbols $U_L,\ U_{0B},\ U_{0T},\ U_R$

C_4 is the free left $\mathbb{Z}\pi_1 X$-module on the symbols $W_L,\ W_R$.

The boundary operators are

$$\begin{cases} \partial_1 A &= (a-1)\,P \\ \partial_1 X &= (x-1)\,P \\ \partial_1 Y &= (x-1)\,P \\ \partial_1 Z &= (x-1)\,P \\ \partial_1 U &= (x-1)\,P \\ \partial_1 V &= (x-1)\,P \end{cases} \qquad \begin{cases} \partial_2 R_{L1} &= Y-U \\ \partial_2 R_{L2} &= X-U \\ \partial_2 R_{0B} &= Y-Z \\ \partial_2 R_{00} &= (1+a-ax)\,(Y-Z) \\ \partial_2 R_{0T} &= Y-Z \\ \partial_2 R_{R1} &= (1-x-xa)\,A + aX - V \\ \partial_2 R_{R2} &= Z-V \end{cases}$$

$$\begin{cases} \partial_3 U_L &= R_{0T} - R_{0B} \\ \partial_3 U_{0B} &= (1+a-ax)\, R_{0B} - R_{00} \\ \partial_3 U_{0T} &= (1+a-ax)\, R_{0T} - R_{00} \\ \partial_3 U_R &= R_{0T} - R_{0B} \end{cases}$$

$$\begin{cases} \partial_4 W_L &= U_{0B} + (1+a-ax)\, U_L - U_{0T} \\ \partial_4 W_R &= U_{0B} + (1+a-ax)\, U_R - U_{0T} \end{cases}$$

The left $\mathbb{Z}\pi_1 X$-module $\mathbb{Z}_2 X$ of 2-cycles can be computed using [6, Hint, p. 383]. From this it follows that:

$$\pi_2 X = 0 \ .$$

Step 5: Since $\pi_2 X = 0$, $\pi_3 X = H_3 C\pi_1 \underline{\underline{\mathbf{X}}}$. From the above chain complex, it follows that:

$$\pi_3 X = (\partial W_\square \ : \ (1+a-ax)\, \partial W_\square = 0) \ .$$

7. CONCLUSION

We have pointed out that every 5-D knot $(S^5,\ kM^3)$ can be concretely visualized by a 3 key frame $X_{-1,0}$, X_{00}, X_{10} representation thus solving problem 1a. Using this method to visualize a 5-D knot, we have shown, using the computing schema of Section V, how to compute $\pi_2 X$ and in some cases $\pi_3 X$ of its complement X, thus solving problem 1b.

The methods given in this paper could be used to:

(1) Compute the k-invariant lying in $H^3\,(\pi_1 X;\ \pi_2 X)$.

(2) Compute the Rohlin invariant μ of 3-manifolds.

(3) Search for homology 3-spheres.

8. APPENDIX: A GENERAL ASPHERICAL DECOMPOSITION

An aspherical decomposition was given in section VI that was tailored to a specific 3-knot. We now show how to construct an aspherical decomposition of X based on the 3 key frame representation in section IV that applies to all 3-knots in general. Figure 19 below should be of help to the reader in understanding this decomposition.

Divide X into three pieces X_L, $X_{\mathbf{0}}$, X_R by cutting along X_{-1} and X_{+1}. Thus,

$$\begin{cases} X & = & X_L \cup X_{\mathbf{0}} \cup X_R \\ X_1 & = & X_L \cup X_{\mathbf{0}} \\ X_{-1} & = & X_R \cup X_{\mathbf{0}} \\ \infty & = & X_L \cap X_R \ , \end{cases}$$

where "∞" denotes the point at infinity. Both X_L and X_R are aspherical since they collapse to the wedge of 1-spheres. Unfortunately, $X_{\mathbf{0}}$ is not in general aspherical, and hence, must be further subdivided.

We continue by cutting $X_{\mathbf{0}}$ along $X_{\mathbf{00}}$ into two pieces X_T and X_B so that

$$\begin{cases} X_{\mathbf{0}} & = & X_T \cup X_B \\ X_{\mathbf{00}} & = & X_T \cap X_B \ . \end{cases}$$

As a result, X is decomposed into a collection $\underline{\underline{\mathbf{X}}}$ consisting of the following pieces:

$$\underline{\underline{\mathbf{X}}} = \{ \ X_L, \ X_R, \ X_T, \ X_B, \ X_{LT}, \ X_{LB}, \ X_{TB}, \ X_{RT}, \ X_{RB}, \ X_{LTB}, \ X_{RTB}, \ \infty \ \} \ ,$$

where by convention the sunscripts satisfy the following rules:

a) $X_\alpha \cap X_\beta = X_{\alpha\beta}$ (for all subscripts α and β)

b) $\alpha\alpha = \alpha$ (for all subscripts α)

b) $\alpha\beta = \beta\alpha$ (for all subscripts α and β) .

(Hence, $X_{TB} = X_{\mathbf{00}}$.)

Remark 6. *The reader should note that:*

$$\infty = X_{LR} = X_{LRT} = X_{LRB} = X_{LRTB} \ .$$

Both X_{LTB} and X_{RTB} are complements of connected knotted graphs in S^3; and X_{TB} collapses to the complement of a connected knotted graph in S^3. Thus, these pieces are aspherical. The one point piece ∞ is obviously aspherical. Finally, all the remaining ellements of $\underline{\underline{\mathbf{X}}}$ are aspherical since each collapses to the wedge of 1-spheres. Hence, $\underline{\underline{\mathbf{X}}}$ is an aspherical structure on X. (See Fig. 20.)

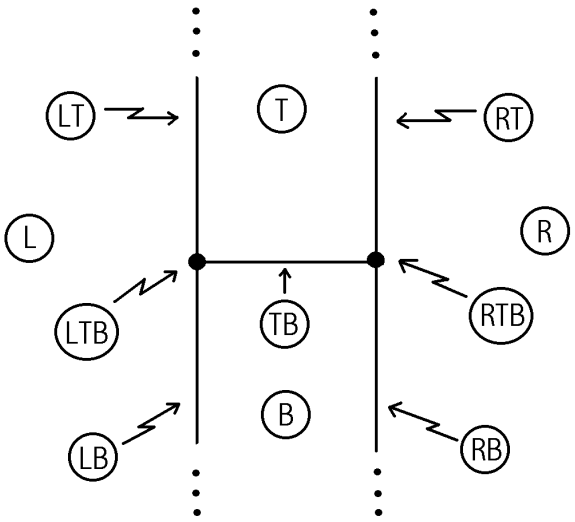


Figure 19. A representation of the general aspherical decomposition of the complement of a 3-knot.

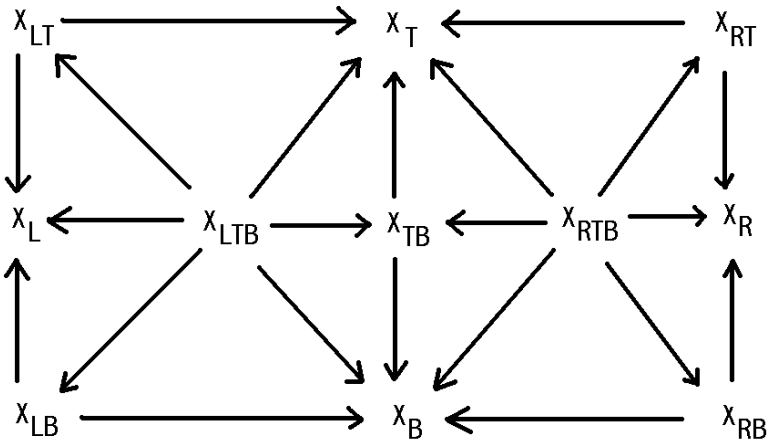


Figure 20. Another representation of the general aspherical decomposition of the complement of a 3-knot. Arrors denote inclusions. For clarity, the piece ∞ together

with its arrows is not shown.

REFERENCES

[1] Crowell, Richard H., **On the Van Kampen Theorem**, Pacific J. Math., 9, No. 1 (1959), 43-50.

[2] Crowell, Richard H. and Fox, Ralph H., "Introduction to Knot Theory," Springer-Verlag, New York, 1977.

[3] Fox, Ralph H., **A Quick Trip Through Knot Theory**, in "Topology of 30manifolds and Related Topics," edited by M.K. Fort, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, (1962), 120-167.

[4] Hemple, John, "3-Manifolds," Annals of Math. Series, No. 86, Princeton University Press, Princeton, New Jersey, 1976.

[5] Lomonaco, S.J., Jr., **The Homology of Group Systems with Applications to Low Dimensional Topology**, AMS Bull., 3, No. 3 (1980), 1049-1052.

[6] Lomonaco, S.J., Jr., **The Homotopy Groups of Knots I. How to Compute the Algebraic 3-type**, Pacific J. Math., Vol. 95, Mo. 2, (1981), 349-390.

[7] Lomonaco, S.J., Jr., **The Third Homotopy Group of Some Higher Dimensional Knots**, in "Knots, Groups, and 3-Manifolds," Princeton University Press, Princeton, New Jersey, 1975, pp. 35 - 45.

[8] Lomonaco, S.J., Jr., **The Homology of Group Systems**, AMS Bull., Vol. 3, No. 3, (1980), pp. 1049 - 1052.

[9] Milnor, J., "Morse Theory," Annals of Math. Studies, No. 51, Princeton University Press, Princeton, New Jersey, 1963.

[10] Abbot, Edwin A., "Flatland," Barnes & Noble, New York, New York, 1963.