MATH221

quiz #4, 12/2/14

Total 100

Solutions

Show all work legibly. **Name**:________________________

1. (20) Let $A = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 1 & 3 \\ 1 & 5 & 8 \end{bmatrix}$. Compute $|A|$ determinant of A.

Solution.

$$\begin{vmatrix} 2 & 0 & -4 \\ 0 & 1 & 3 \\ 1 & 5 & 8 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 1 & 5 & 8 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 5 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & -5 \end{vmatrix} = 2(-5)\begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{vmatrix} = -10.$$

$|A| =$

2. (80) Let $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$.

(a) **(20)** Find the eigenvalues λ_1 and λ_2 of A.

Solution. $0 = \det(A - \lambda I) = \lambda(\lambda - 5)$, hence $\lambda_1 = 0$, $\lambda_2 = 5$.

The eigenvalues of A are: $\lambda_1 =$ $\lambda_2 =$

(b) **(20)** Find unit norm eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of A.

Solution. For $\lambda = 0$ and $(A - \lambda I)\mathbf{x} = 0$ one has $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. If eigenvector of magnitude one is needed, then $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$.

For $\lambda = 5$ and $(A - \lambda I)\mathbf{x} = 0$ one has $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. If eigenvector of magnitude one is needed, then $\mathbf{v}_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$.

The eigenvectors of A are: $\mathbf{v}_1 =$ $\mathbf{v}_2 =$

(c) **(20)** Find a matrix V such that $V^T A V = \Lambda$, where Λ is a diagonal matrix.

Solution.

$$V = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

$V =$

(d) **(20)** Compute A^6.

Solution. Since $A = V\Lambda V^{-1}$ one has

$$A^6 = V\Lambda^6 V^{-1} = \begin{bmatrix} 4 \cdot 5^5 & 2 \cdot 5^5 \\ 2 \cdot 5^5 & 5^5 \end{bmatrix} = \begin{bmatrix} 12500 & 6250 \\ 6250 & 3125 \end{bmatrix}.$$

$A^6 =$

3. (20) Let $\mathbf{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Compute the distance d from $\mathbf{u}$ to the line through $\mathbf{y}$ and the origin.

Solution. Projection of $\mathbf{u}$ onto the line is $\frac{\mathbf{u}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} = 3\mathbf{y}$. The squared distance is $d^2 = (\mathbf{u} - 3\mathbf{y})^T(\mathbf{u} - 3\mathbf{y}) = 5$.

$d = \sqrt{5}$.