MATH221

quiz #4, 12/03/13

Total 100

Solutions

Show all work legibly. **Name**:________________________

1. (20) Find the characteristic polynomial $\mathbf{p}(x)$ of the matrix $A = \begin{bmatrix} 6 & 2 \\ 21 & 7 \end{bmatrix}$.

   Solution. $\mathbf{p}(x) = (\lambda - 6)(\lambda - 7) - 42 = \lambda(\lambda - 13)$

   $\mathbf{p}(x) =$

2. (20) Find the the eigenvalues and the corresponding eigenvectors of the matrix $A = \begin{bmatrix} 6 & 2 \\ 21 & 7 \end{bmatrix}$.

   Solution. The roots of $0 = \lambda(\lambda - 13)$ are $\lambda_1 = 0$ and $\lambda_2 = 13$.

   When $\lambda = 0$ the solution of $\begin{bmatrix} 6 & 2 \\ 21 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is, for example, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

   When $\lambda = 13$ the solution of $\begin{bmatrix} -7 & 2 \\ 21 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is, for example, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$.

3. (20) Find the inverse V^{-1} of the matrix $V = [\mathbf{v}_1, \mathbf{v}_2] = \begin{bmatrix} 1 & 2 \\ -3 & 7 \end{bmatrix}$.

   Solution.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 13 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/13 & 1/13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7/13 & -2/13 \\ 0 & 1 & 3/13 & 1/13 \end{bmatrix}.$$

   Hence $V^{-1} = \frac{1}{13}\begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix}$.

4. (20) Compute A^{10} for $A = \begin{bmatrix} 6 & 2 \\ 21 & 7 \end{bmatrix}$.

   Solution. Since $A = VDV^{-1}$ one has

$$A^{10} = VD^{10}V^{-1} = \frac{1}{13}\begin{bmatrix} 1 & 2 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 13^{10} \end{bmatrix}\begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix} = 13^9\begin{bmatrix} 6 & 2 \\ -21 & -7 \end{bmatrix}.$$

   $A =$

5. (20) Let L be a line in $\mathbf{R}^2$ passing through the points $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(a) (20) Wright a parametric equation for the line L.

Solution. $L = \{\mathbf{a} + t\mathbf{u} \,:\, -\infty < t < \infty\}$, where $\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{b} - \mathbf{a})$ is a unit norm vector.

$L =$

(b) (20) Find the projection $\mathbf{p}$ of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ on L.

Solution. $L = \{\mathbf{a} + t\mathbf{u} \,:\, -\infty < t < \infty\}$, where $\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{b} - \mathbf{a})$ is a unit norm vector. The projection $\mathbf{p} = \mathbf{a} + c\mathbf{u}$, and $\mathbf{v} - \mathbf{p}$ is orthogonal to to $\mathbf{u}$. That is $0 = \mathbf{u}^T(\mathbf{v} - \mathbf{p}) = \mathbf{u}^T(\mathbf{v} - \mathbf{a} - c\mathbf{u}) = \mathbf{u}^T(\mathbf{v} - \mathbf{a}) - c$, and $c = \mathbf{u}^T(\mathbf{v} - \mathbf{a})$. Finally $\mathbf{p} = \mathbf{a} + \mathbf{u}\mathbf{u}^T(\mathbf{v} - \mathbf{a}) = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\mathbf{p} =$