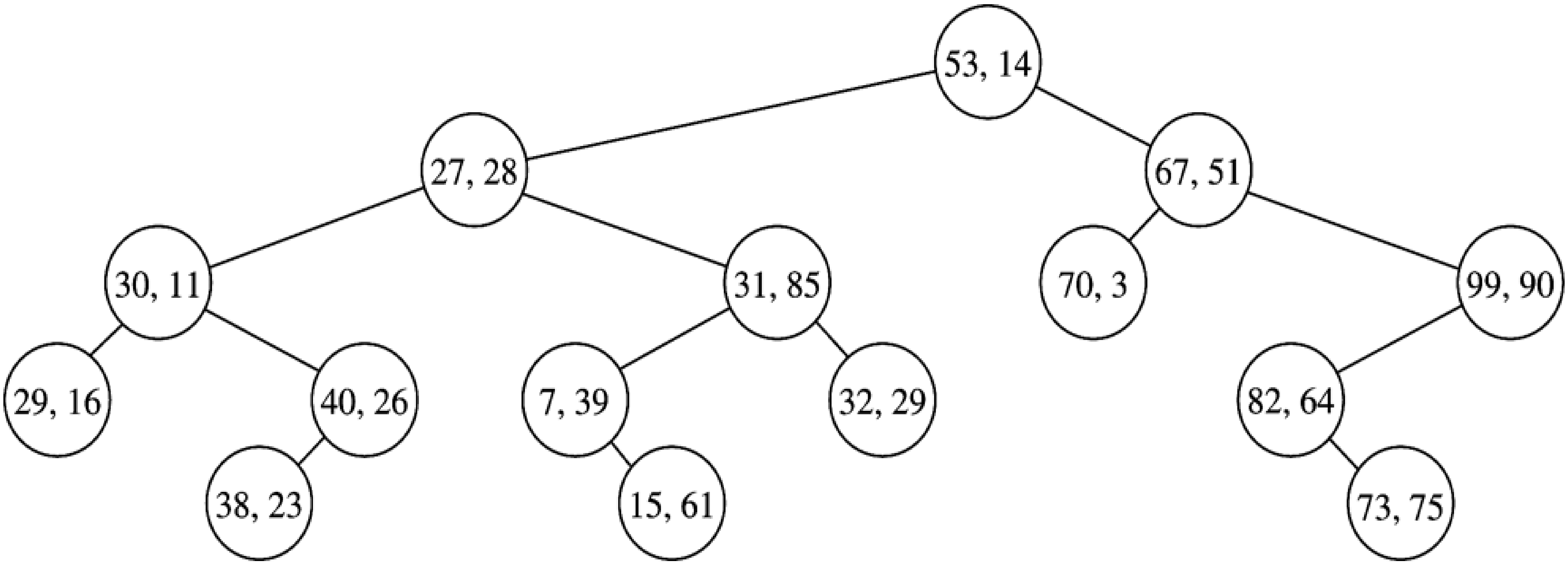
CMSC 341

K-D Trees

K-D Tree

- Introduction
  - Multiple dimensional data
    - Range queries in databases of multiple keys:
      Ex. find persons with
      $34 \le age \le 49$ and $\$100k \le annual\ income \le \$150k$
    - GIS (geographic information system)
    - Computer graphics
  - Extending BST from one dimensional to k-dimensional
    - It is a binary tree
    - Organized by levels (root is at level 0, its children level 1, etc.)
    - Tree branching at level 0 according to the first key, at level 1 according to the second key, etc.
- KdNode
  - Each node has a vector of keys, in addition to the pointers to its subtrees.

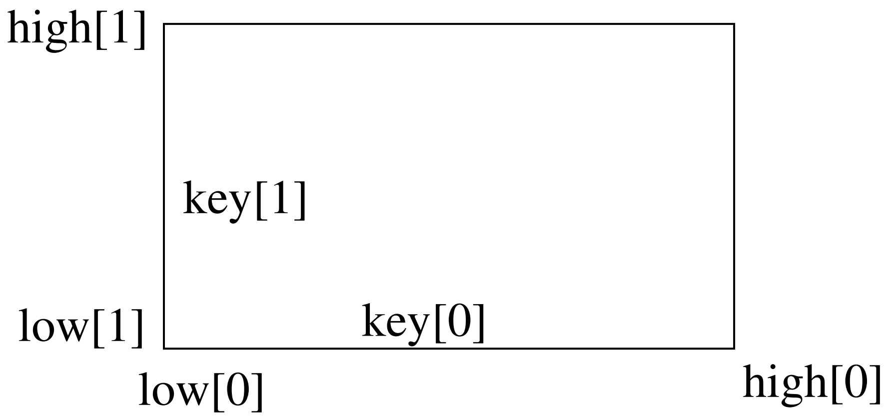
K-D Tree



- A 2-D tree example

2-D Tree Operations

- Insert
  - A 2-D item (vector of size 2 for the two keys) is inserted
  - New node is inserted as a leaf
  - Different keys are compared at different levels
- Find/print with an orthogonal (rectangular) range



  - exact match: insert (low[level] = high[level] for all levels)
  - partial match: (query ranges are given to only some of the k keys, other keys can be thought in range $\pm\infty$)

2-D Tree Insertion

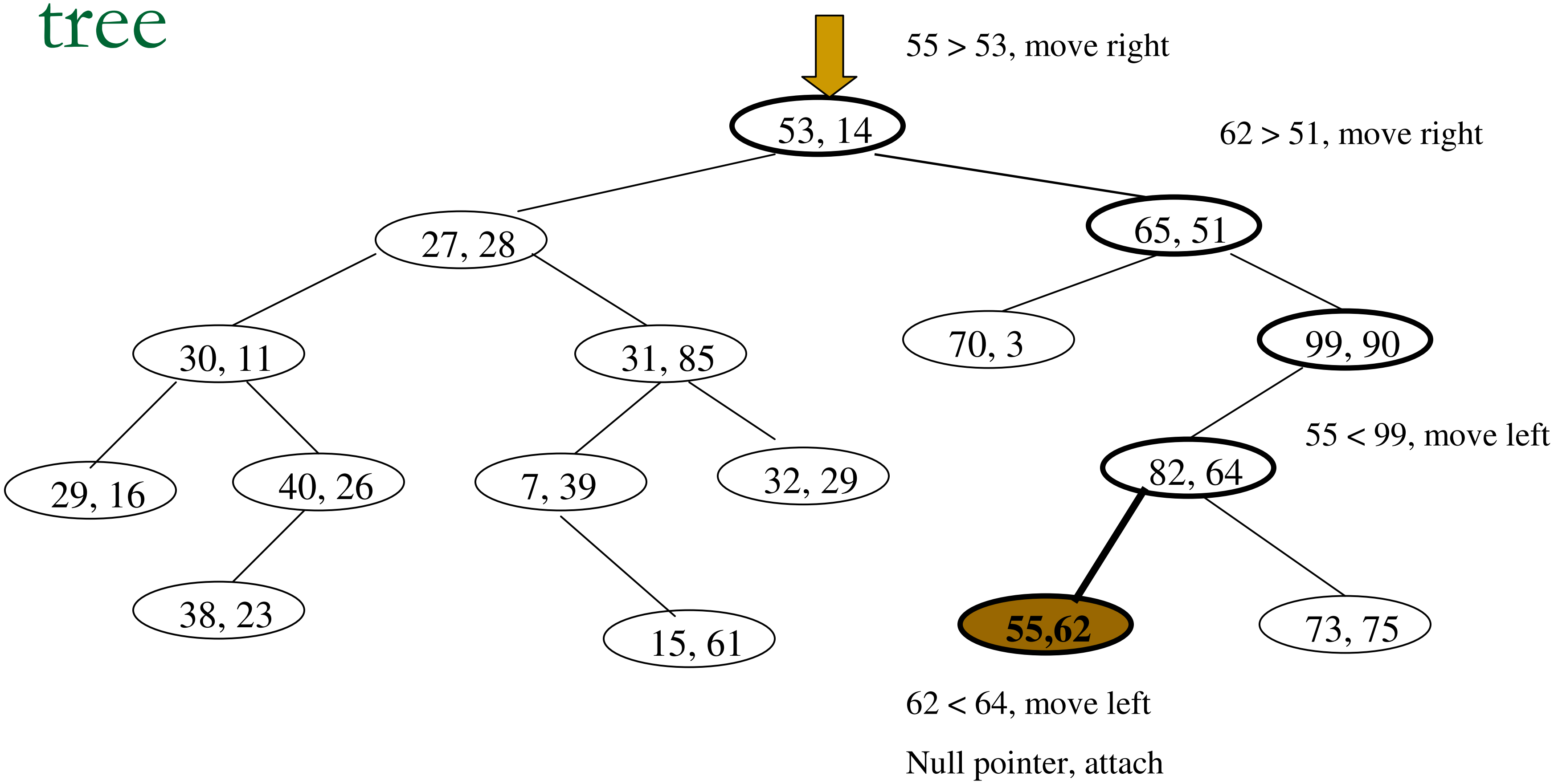
```
public void insert(Vector <T> x)
{
      root = insert( x, root, 0);
}

// this code is specific for 2-D trees
private KdNode<T> insert(Vector <T> x, KdNode<T> t, int level)
{
      if (t == null)
         t = new KdNode(x);

      int compareResult = x.get(level).compareTo(t.data.get(level));
      if (compareResult < 0)
         t.left = insert(x, t.left, 1 - level);
      else if( compareResult > 0)
         t.right = insert(x, t.right, 1 - level);
      else
              ;  // do nothing if equal

      return t;
}
```

Insert (55, 62) into the following 2-D tree



2-D Tree: printRange

```
/**
 * Print items satisfying
 * lowRange.get(0) <= x.get(0) <= highRange.get(0)
 * and
 * lowRange.get(1) <= x.get(1) <= highRange.get(1)
 */
public void printRange(Vector <T> lowRange,
                                Vector <T>highRange)
{
    printRange(lowRange, highRange, root, 0);
}
```

2-D Tree: printRange (cont.)

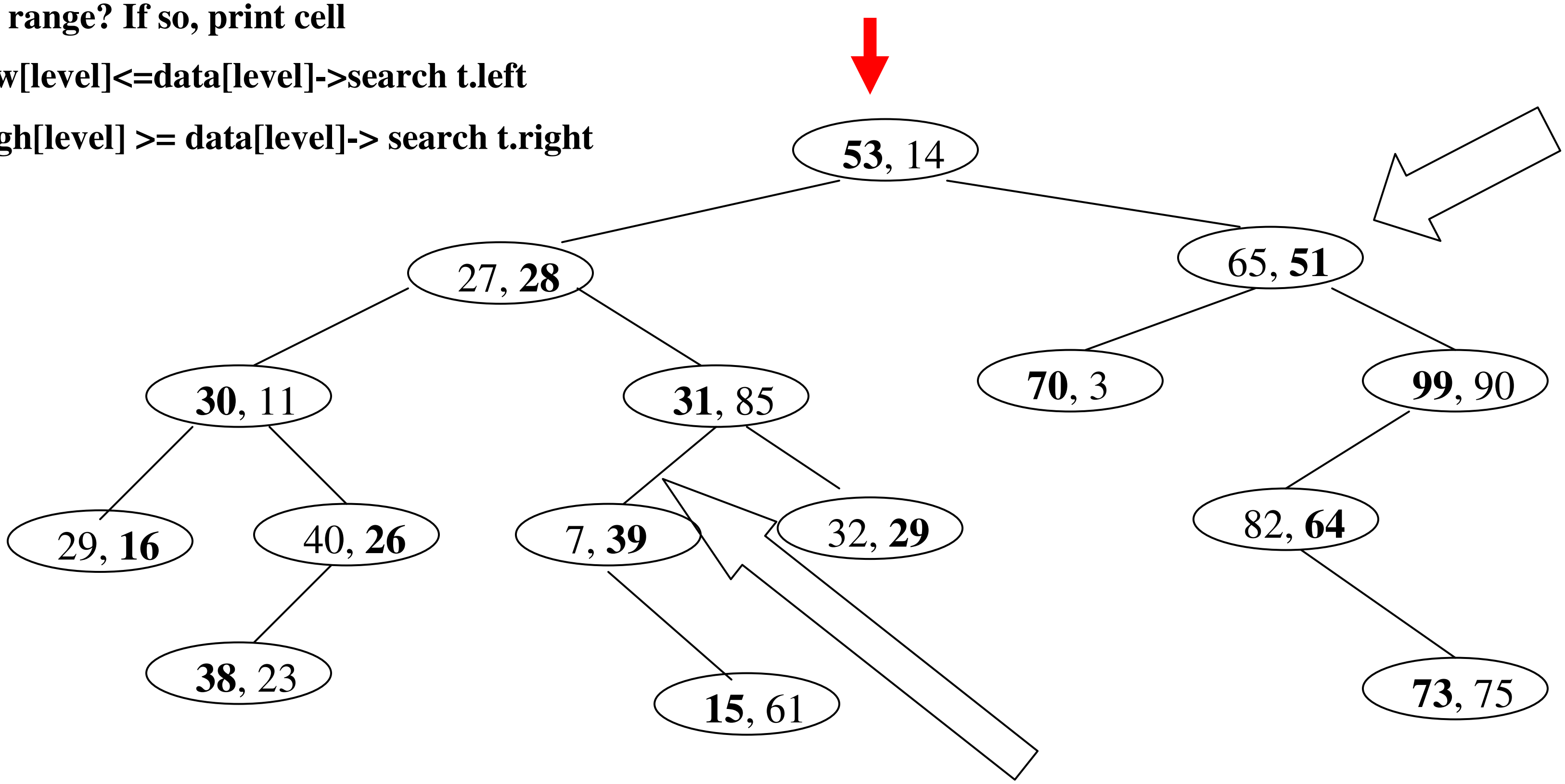
```
private void
printRange(Vector <T> low,Vector <T> high,
                                     KdNode<T> t, int level)
{
   if (t != null)
   {
      if ((low.get(0).compareTo(t.data.get(0)) <= 0  &&
               t.data.get(0).compareTo(high.get(0)) <=0)
         &&(low.get(1).compareTo(t.data.get(1)) <= 0 &&
               t.data.get(1).compareTo(high.get(1)) <= 0))
        System.out.println("(" + t.data.get(0) + "," +
                                      t.data.get(1) + ")");
      if (low.get(level).compareTo(t.data.get(level)) <= 0)
               printRange(low, high, t.left, 1 - level);
      if (high.get(level).compareTo(t.data.get(level)) >= 0)
                printRange(low, high, t.right, 1 - level);
   }
}
```

printRange in a 2-D Tree

In range? If so, print cell

low[level]<=data[level]->search t.left

high[level] >= data[level]-> search t.right



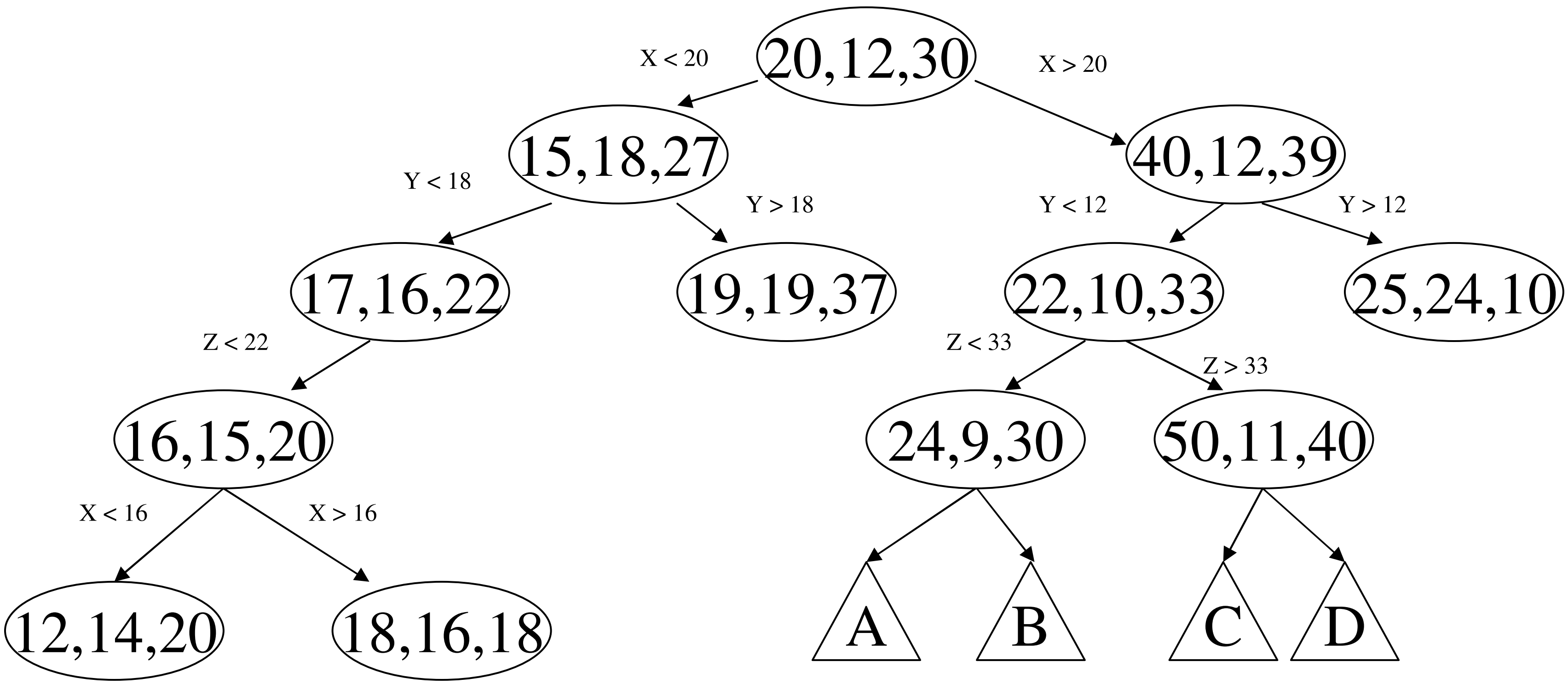
low[0] = 35, high[0] = 40;

low[1] = 23, high[1] = 30;

This sub-tree is never searched.

Searching is "preorder". Efficiency is obtained by "pruning" subtrees from the search.

3-D Tree example



What property (or properties) do the nodes in the subtrees labeled A, B, C, and D have?

K-D Operations

- Modify the 2-D insert code so that it works for K-D trees.
- Modify the 2-D printRange code so that it works for K-D trees.

K-D Tree Performance

- Insert
  - Average and balanced trees: O(lg N)
  - Worst case: O(N)
- Print/search with a square range query
  - Exact match: same as insert (low[level] = high[level] for all levels)
  - Range query: for M matches
    - Perfectly balanced tree:

      K-D trees: $O(M + kN^{(1-1/k)})$

      2-D trees: $O(M + \sqrt{N})$
    - Partial match

      in a random tree: $O(M + N^{\alpha})$ where $\alpha = (-3 + \sqrt{17}) / 2$

K-D Tree Performance

- More on range query in a perfectly balanced 2-D tree:
  - Consider one boundary of the square (say, low[0])
  - Let T(N) be the number of nodes to be looked at with respect to low[0]. For the current node, we may need to look at
    - One of the two children (e.g., node (27, 28), and
    - Two of the four grand children (e.g., nodes (30, 11) and (31, 85).
  - Write T(N) = 2 T(N/4) + c, where N/4 is the size of subtrees 2 levels down (we are dealing with a perfectly balanced tree here), and c = 3.
  - Solving this recurrence equation:

  $T(N) = 2T(N/4) + c = 2(2T(N/16) + c) + c$

  …

  $= c(1 + 2 + \cdots + 2^{\wedge}(\log_4 N) = 2^{\wedge}(1+ \log_4 N) – 1$

  $= 2*2^{\wedge}(\log_4 N) – 1 = 2^{\wedge} ((\log_2 N)/2) – 1 = O(\sqrt{N})$

K-D Tree Remarks

- Remove
  - No good remove algorithm beyond lazy deletion
    (mark the node as removed)
- Balancing K-D Tree
  - No known strategy to guarantee a balanced 2-D tree
  - Periodic re-balance
- Extending 2-D tree algorithms to k-D
  - Cycle through the keys at each level