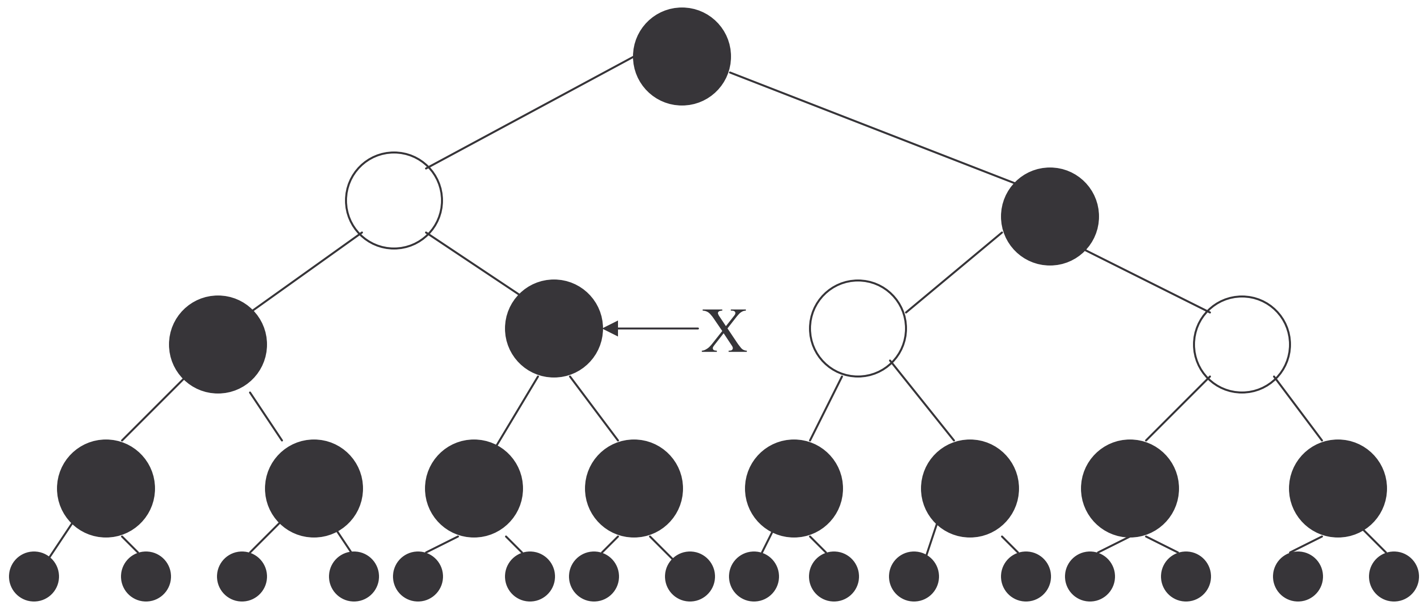
Red-Black Trees

Definitions
and
Bottom-Up Insertion

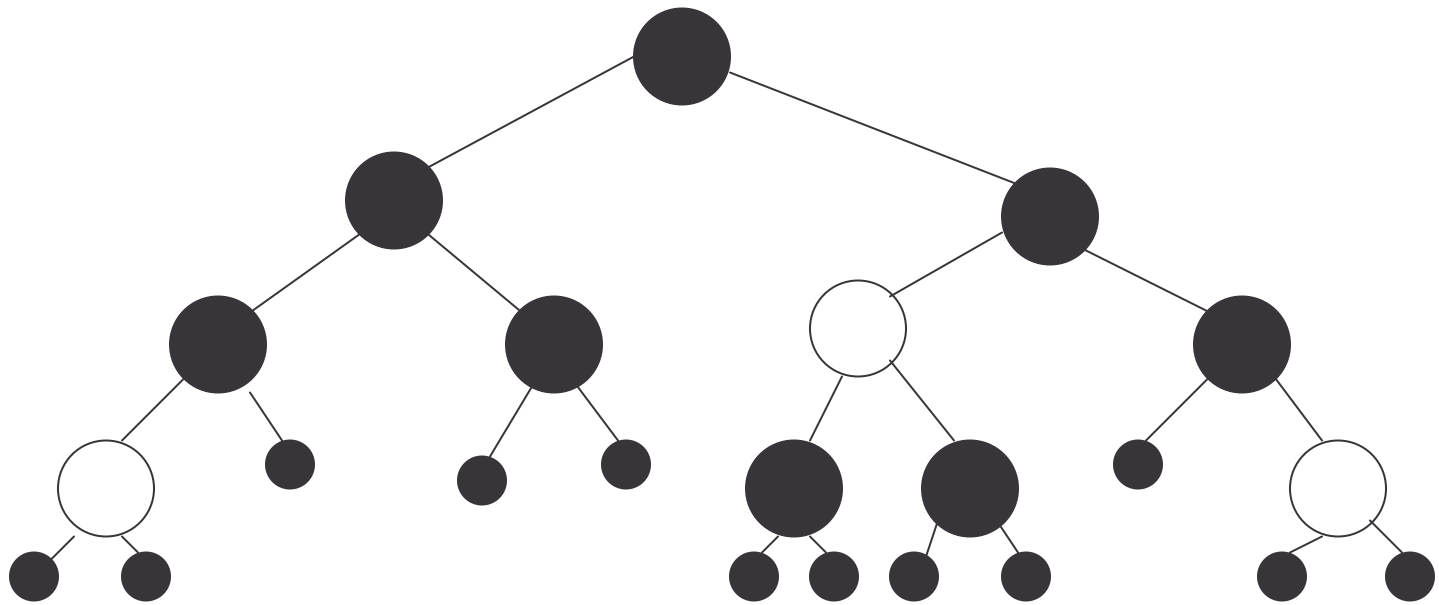
Red-Black Trees

- Definition: A red-black tree is a binary search tree in which:
  - Every node is colored either Red or Black.
  - Each NULL pointer is considered to be a Black "node".
  - If a node is Red, then both of its children are Black.
  - Every path from a node to a NULL contains the same number of Black nodes.
  - By convention, the root is Black
- Definition: The black-height of a node, X, in a red-black tree is the number of Black nodes on any path to a NULL, not counting X.



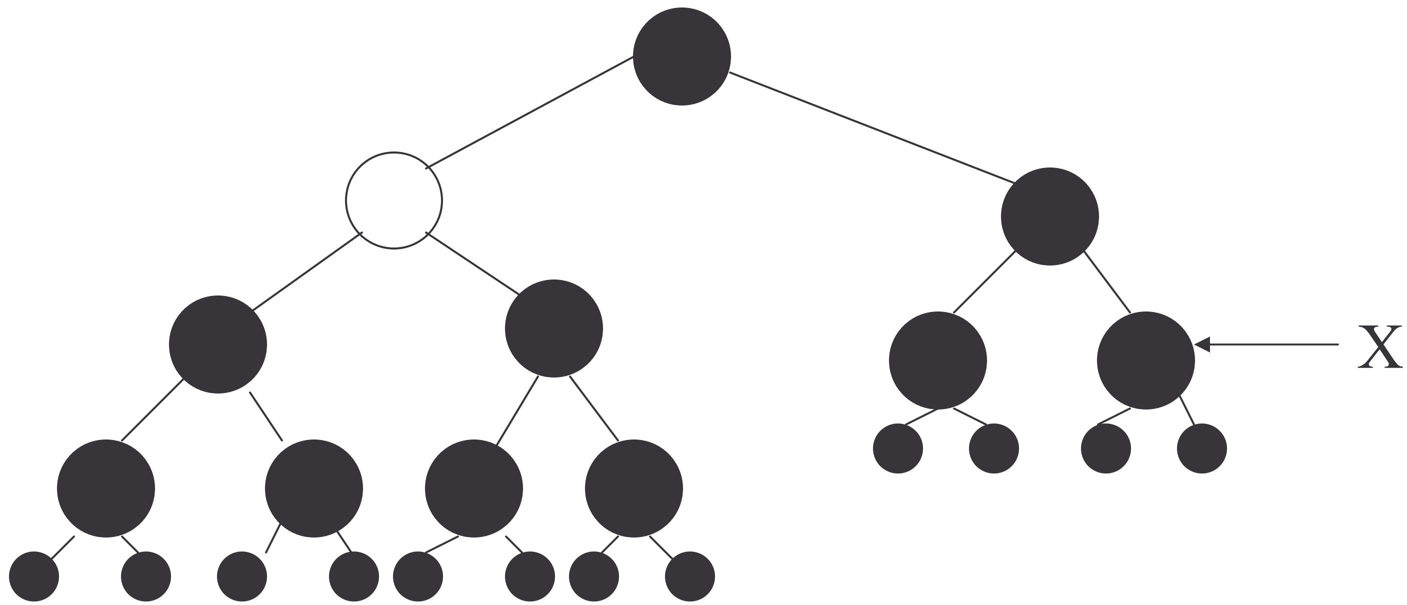
A Red-Black Tree with NULLs shown

Black-Height of the tree (the root) = 3
Black-Height of node “X” = 2



A Red-Black Tree with

Black-Height = 3



Black Height of the tree?

Black Height of X?

Theorem 1 – Any red-black tree with root $\boldsymbol{x}$, has $\mathbf{n \geq 2^{bh(x)} - 1}$ nodes, where bh(x) is the black height of node x.

Proof: by induction on height of x.

Theorem 2 – In a red-black tree, at least half the nodes on any path from the root to a NULL must be Black.

Proof – If there is a Red node on the path, there must be a corresponding Black node.

Algebraically this theorem means

$$bh(x) \geq h/2$$

Theorem 3 – In a red-black tree, no path from any node, X, to a NULL is more than twice as long as any other path from X to any other NULL.

Proof: By definition, every path from a node to any NULL contains the same number of Black nodes. By Theorem 2, a least ½ the nodes on any such path are Black. Therefore, there can no more than twice as many nodes on any path from X to a NULL as on any other path. Therefore the length of every path is no more than twice as long as any other path.

Theorem 4 –

A red-black tree with ***n*** nodes has height

$$h \leq 2 \lg(\boldsymbol{n} + 1).$$

Proof: Let h be the height of the red-black tree with root x. By Theorem 2,

$$bh(x) \geq h/2$$

From Theorem 1, $\boldsymbol{n} \geq 2^{bh(x)} - 1$

Therefore $n \geq 2^{h/2} - 1$

$$n + 1 \geq 2^{h/2}$$

$$\lg(n + 1) \geq h/2$$

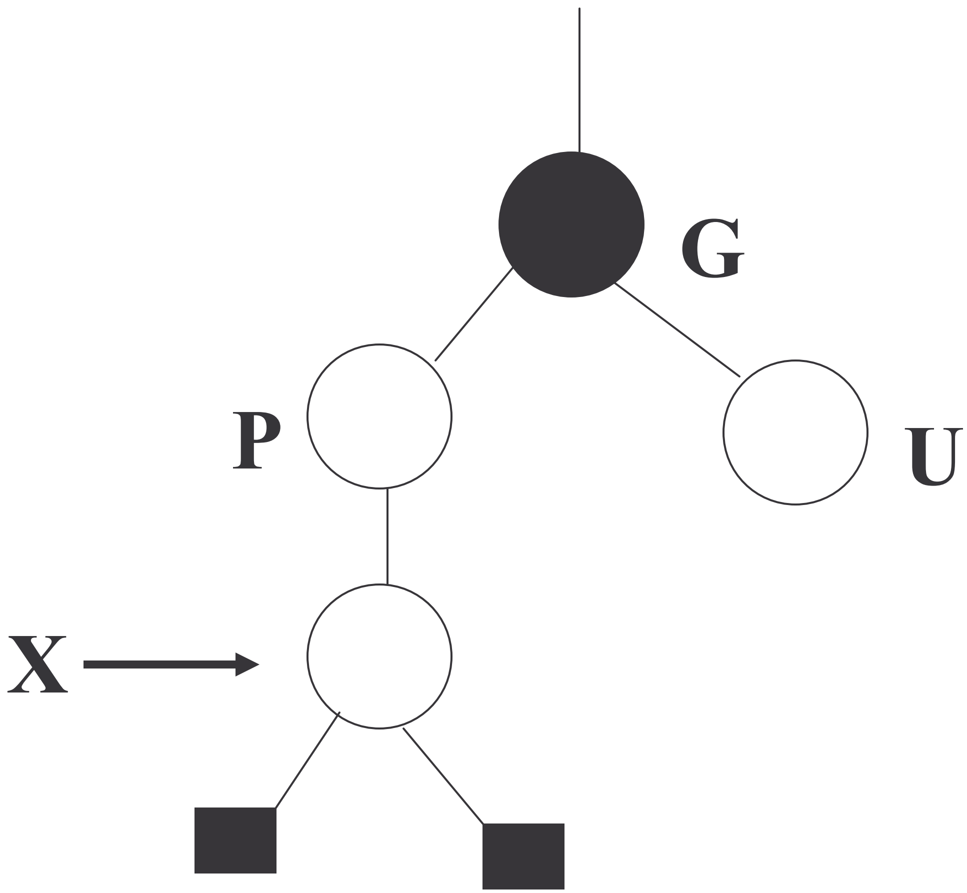
$$2\lg(n + 1) \geq h$$

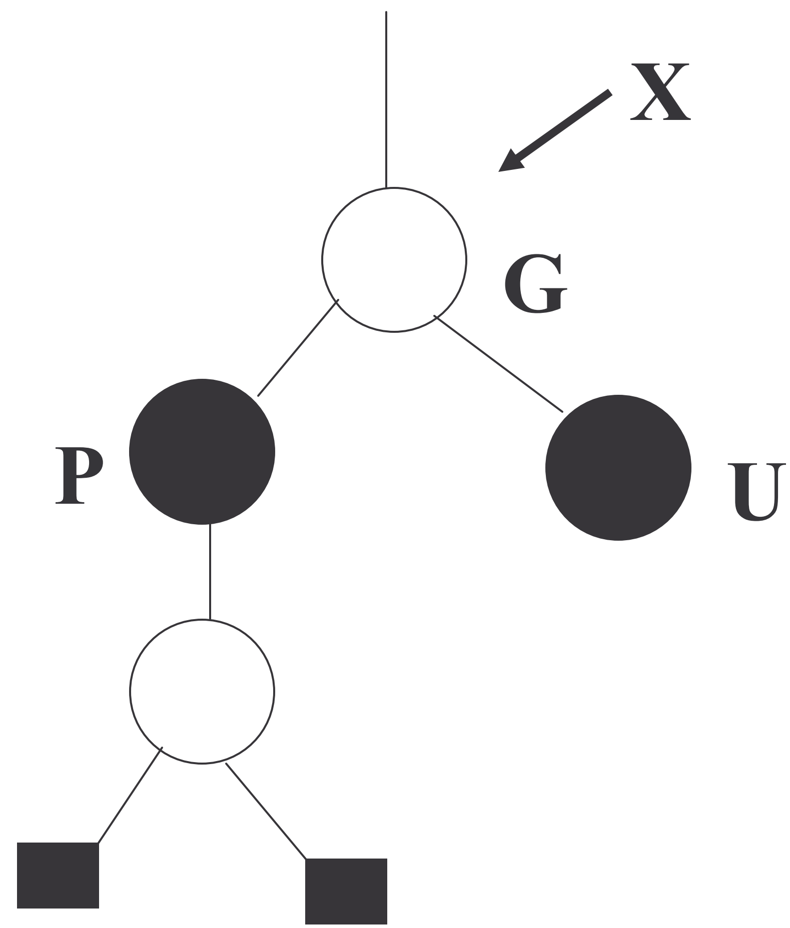
Bottom –Up Insertion

- Insert node as usual in BST
- Color the node Red
- What Red-Black property may be violated?
  - Every node is Red or Black?
  - NULLs are Black?
  - If node is Red, both children must be Black?
  - Every path from node to descendant NULL must contain the same number of Blacks?

Bottom Up Insertion

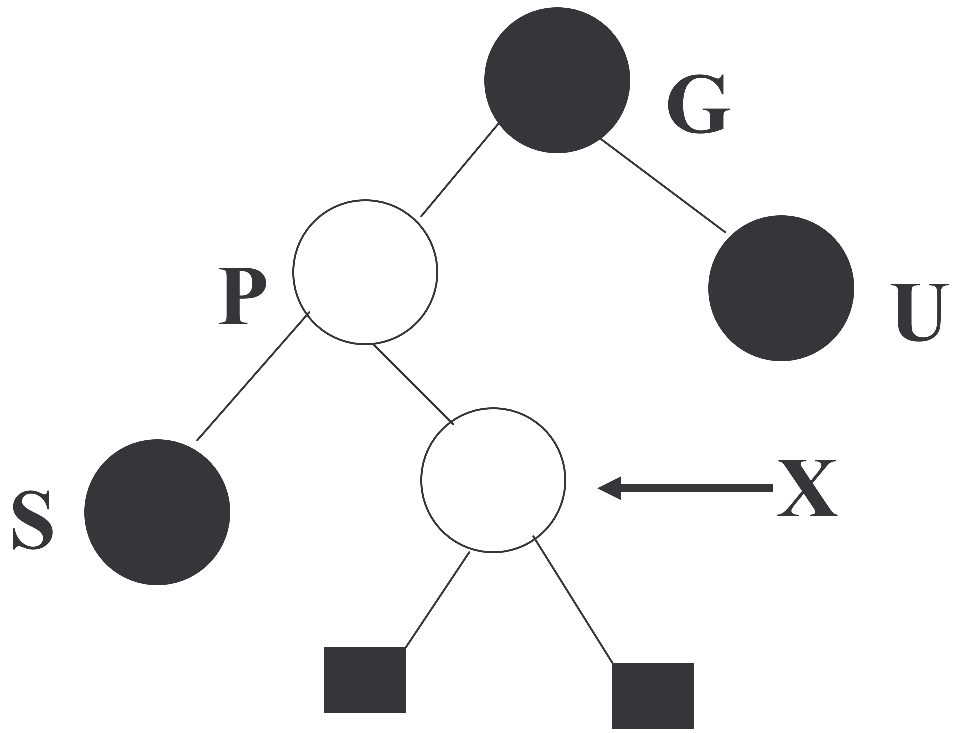
- Insert node; Color it Red; X is pointer to it
- Cases
  - 0: X is the root -- color it Black
  - 1: Both parent and uncle are Red -- color parent and uncle Black, color grandparent Red. Point X to grandparent and check new situation.
  - 2 (zig-zag): Parent is Red, but uncle is Black. X and its parent are opposite type children -- color grandparent Red, color X Black, rotate left(right) on parent, rotate right(left) on grandparent
  - 3 (zig-zig): Parent is Red, but uncle is Black. X and its parent are both left (right) children -- color parent Black, color grandparent Red, rotate right(left) on grandparent

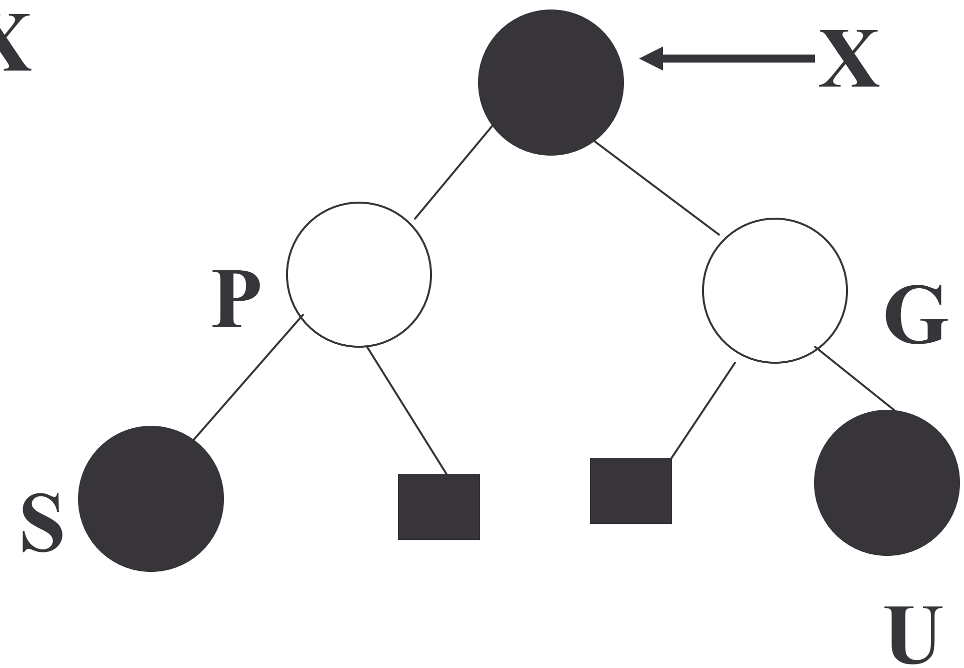




Case 1 – U is Red

Just Recolor and move up

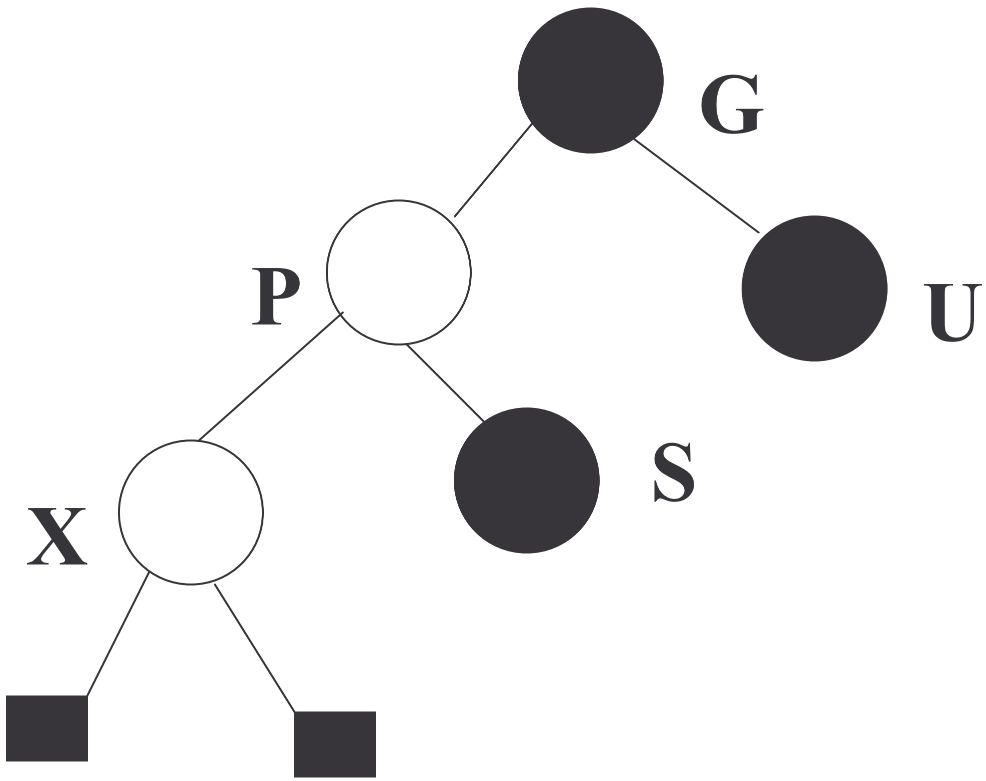




Case 2 – Zig-Zag

Double Rotate
  X around P; X around G

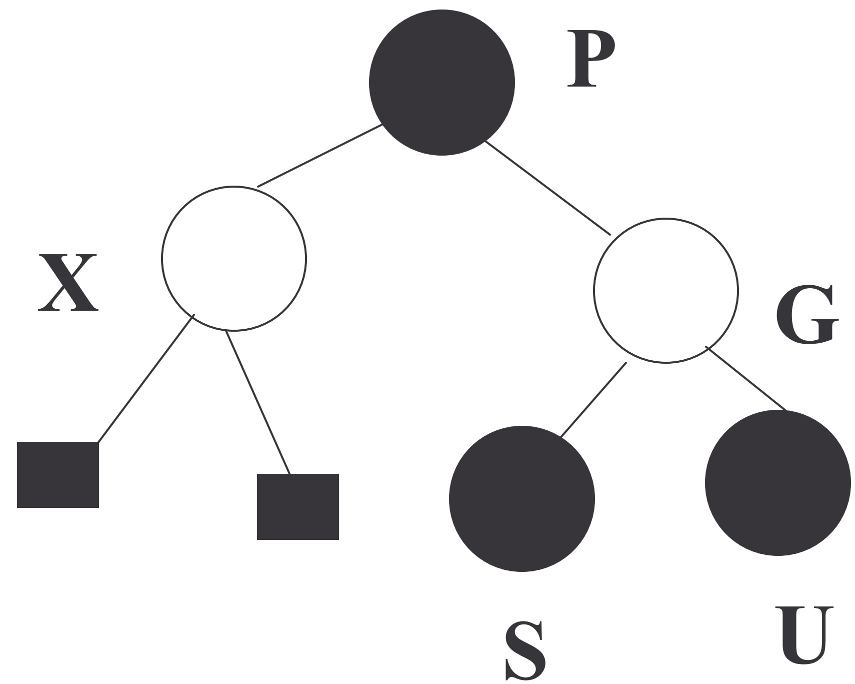
Recolor G and X



Case 3 – Zig-Zig

Single Rotate P around G

Recolor P and G



Asymptotic Cost of Insertion

- $O(\lg n)$ to descend to insertion point
- $O(1)$ to do insertion
- $O(\lg n)$ to ascend and readjust == worst case only for case 1

- Total: $O(\log n)$

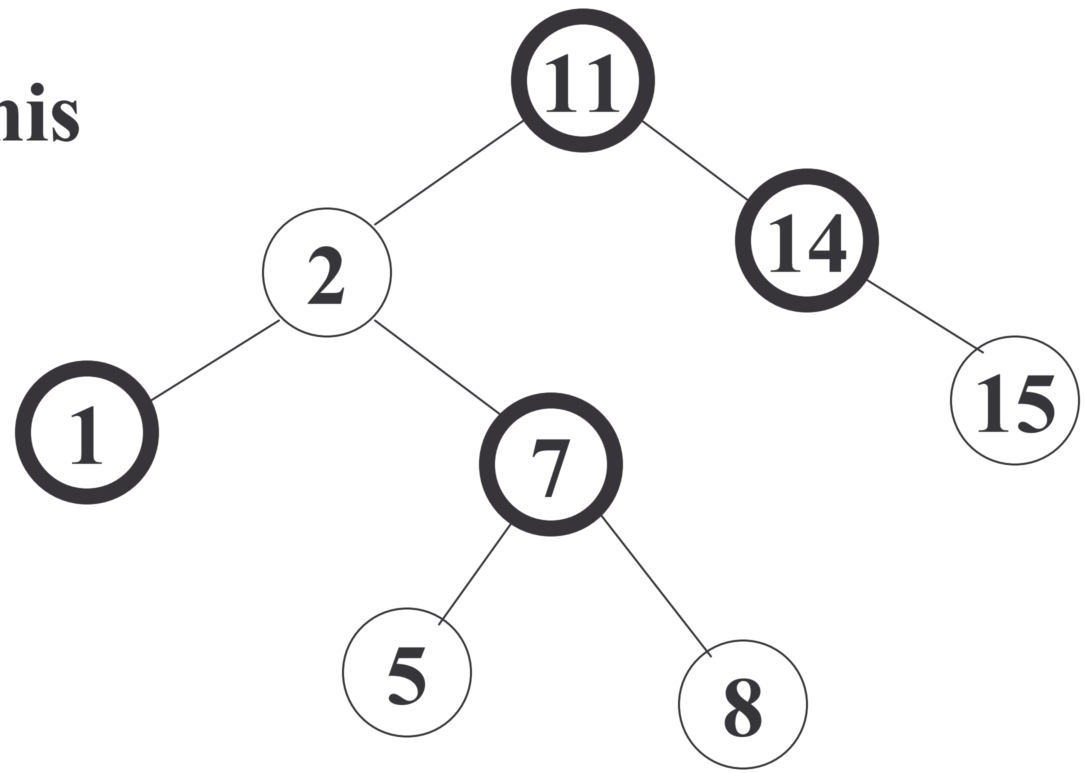
Top-Down Insertion

An alternative to this “bottom-up” insertion is “top-down” insertion.

Top-down is iterative. It moves down the tree, “fixing” things as it goes.

What is the objective of top-down’s “fixes”?

Insert 4 into this R-B Tree



Insertion Practice

Insert the values 2, 1, 4, 5, 9, 3, 6, 7 into an initially empty Red-Black Tree