# Informal Seminar Notes on Lie Groups©

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Abstract. The objective of these informal notes is to communicate in an informal concise fashion that part of the theory of Lie groups relevant to the understanding of quantum mechanics. No claim is made as to the competeness of these notes.

## 1. Introduction.

We begin with the definition of a topological group.

**Definition 1.** *A set $G$ is a* ***topological group*** *if*

1. *$G$ is an abstract group*


[†]Partially supported by the L-O-O-P Fund.

2. *G is a topological space*
3. *The group operations of G are continuous, i.e.*
   (a) *The multiplication map*
   $$\begin{aligned} G \times G &\longrightarrow G \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned}$$
   *is continuous.*
   (b) *The inverse map*
   $$\begin{aligned} G &\longrightarrow G \\ g &\mapsto g^{-1} \end{aligned}$$
   *is continuous.*

**Definition 2.** *A **Lie group** $G$ is*

1. *A $C^\omega$ (i.e., real analytic) manifold*
2. *A topological group*

*such that the group multiplication and inverse maps are $C^\omega$.*

**Remark 1.** ***Hilberts Fifth Problem*** *(1900) How are Lie's concepts of continuous groups of transformations of manifolds approachable in our investigation without the assumption of differentiability?*

The answer to this question was given by Gleason, Montgomery-Zippin:

**Theorem 1** [Gleason, Montgomery-Zippin]**.** *Every locally euclidean group is a Lie group.*

As a consequence, we can replace each $C^\omega$ in the above definition of a Lie group with a $C^\infty$ .

## 2. Some examples of Lie groups

Let $V$ denote an $n$-dimensional vector space over the real numbers $\mathbb{R}$ with the standard vector inner product which we denote by $\langle\ ,\ \rangle$.

- $GL(n, \mathbb{R})$ The **real general linear group** of all automorphisms of the vector space $V$. This can be identified with the group of all nonsingular $n \times n$ matrices over the reals.
- $O(n)$ The **real orthogonal group** is the group of all automorphisms which preserve the inner product $\langle\ ,\ \rangle$. This can be identified with the group of orthogonal matrices , i.e, matrices $A$ of the form

$$A^T = A^{-1}$$

  where the superscipt "$T$" denotes the matrix transpose.
- $SL(n, \mathbb{R})$ The **real special linear group** is the group of all real $n \times n$ matrices of determinant 1. $SL(n, \mathbb{R})$ is the group of all rigid motions in hyperbolic $n$-space.
- $SO(n) = O(n) \cap SL(n, \mathbb{R})$ The **special orthogonal group** is the group of all orthogonal real $n \times n$ matrices of determinant 1. This group can be identified with the group of all rotations in $\mathbb{R}^n$ about a fixed point such as the origin.

Let $W$ denote an $n$-dimensional vector space over the complex numbers $\mathbb{C}$ with the standard sesquilinear inner product which we also denote by $\langle\ ,\ \rangle$.

- $GL(n, \mathbb{C})$ The **complex general linear group** of all automorphisms of the vector space $W$. This can be identified with the group of all nonsingular $n \times n$ matrices over the complexes.
- $SL(n, \mathbb{C})$ The **complex special linear group** is the group of all complex $n \times n$ matrices of determinant 1.
- $U(n)$ The **unitary group** is the group of all $n \times n$ unitary matrices over the compex numbers $\mathbb{C}$, i.e., all $n \times n$ complex matrices $A$ such that

$$A^\dagger = A^{-1}$$

  where $A^\dagger$ denotes the conjugate transpose.

- $SU(n) = U(n) \cap SL(n, \mathbb{C})$ The special unitary group is the group of all unitary matrices of determinant 1.

The only connected abelian Lie groups are:

- $\mathbb{R}^n$ **Euclidean $n$-space under vector addition**
- $\mathbb{T}^n = U(1) \times U(1) \times \ \ldots \ \times U(1)$ The $n$-**dimensional torus**. Please note that $U(1)$ is the same as the unit circle group $S^1$, i.e., all complex numbers of unit norm.
- And all direct products $\mathbb{R}^m \times \mathbb{T}^n$.

There are many other Lie groups. We mention only four more.

- $O(3, 1)$ The **Lorentz group** which is the group of all automorphisms of $V$ ($n = 4$) which preserve a nondegenerate positive definite bilinear form of signature 2 on $V$, such as
$$\langle x, y \rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$$
- $H(n)$ The **group of orientation preserving euclidean motions** in $n$-dimensional euclidean space. This can be identified with the semidirect product $SO(n) \ltimes \mathbb{R}^n$.
- $Spin(n)$ The **spin(or) group** is the Universal covering group of $SO(n)$. This group can be constructed in terms of Clifford algebras.
- $Sp(n)$ The symplectic group is the group of all $n \times n$ matrices over the quaternians $\mathbb{H}$ such that
$$\overline{A}^T = A^{-1}$$
where $\overline{A}$ denotes the conjugate in the quaterians of each element of $A$, and where the superscript "$T$" denotes the transpose.

## 3. AN EXAMPLE: THE RELATION BETWEEN THE LIE GROUPS $SU(2) = Spin(2)$ AND $SO(3)$

**3.1. $SO(3) = \mathbb{R}P^3$.** Let $S^2$ denote the standard 2-sphere of radius 1, as a riemannian manifold. We identify $S^2$ with the set of all unit length vectors $\overrightarrow{n}$ in $\mathbb{R}^3$ that are based at the origin. $SO(3)$ can be identified with the group of orientation preserving isometries of $S^2$, i.e., with the group of all rotations of $S^2$ about its center. The rotations of $S^2$ can also be identified with the vectors

$$\theta\overrightarrow{n}$$

where $\overrightarrow{n} \in S^2$ denotes the direction of the axis of rotation, and where $\theta$ denotes the angle of rotation. We use the convention that positive $\theta$ correspond to a rotation in which $\theta\overrightarrow{n}$ represents a right handed screw. We next note that the following rotations are the same:

$$\begin{cases} (\theta + 2\pi)\overrightarrow{n} = \theta\overrightarrow{n} \\ \\ (-\theta)\overrightarrow{n} = \theta(-\overrightarrow{n}) \end{cases}$$

Hence,

$$\pi\overrightarrow{n} = \pi(-\overrightarrow{n})$$

From this it follows that $SO(3)$ can be identified with

$$\left\{\theta\overrightarrow{n} \;\mid\; \overrightarrow{n} \in S^2 \text{ and } 0 \le \theta \le \pi\right\}$$

modulo the identifications

$$\pi\overrightarrow{n} = \pi(-\overrightarrow{n}) \quad \forall\overrightarrow{n} \in S^2$$

But this is the same as the standard 3-ball $B^3$ of radius $\pi$ with antipodal points on its boundary $\partial B^3$ identified. Thus, $SO(3)$ can be identified with projective 3-space $\mathbb{R}P^3$.

Thus the fundamental group $\pi_1(SO(3))$ of $SO(3)$ is the cyclic goup of order 2, i.e., $\mathbb{Z}_2$. Any simple closed curve in $SO(3)$ of the form

$$\gamma_{\overrightarrow{n}}(\theta) = \theta\overrightarrow{n}, \text{ where } -\pi \le \theta \le \pi$$

represents the generator of $\pi_1(SO(3))$. The reader should note that $\gamma_{\overrightarrow{n}}$ corresponds to continuously rotating $S^2$ through a full angle $2\pi$ about the axis $\overrightarrow{n}$, i.e., one continuos full rotation about $\overrightarrow{n}$.

## 4. PARAMETRIZATIONS OF $SO(3)$.

**4.1. Euler angles.** Every rotation $A$ in 3-space can be written as the composion of three separate rotations

$$A = R_z(\psi)R_x(\theta)R_z(\varphi)$$

where

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a counterclockwise rotation about the Z-axis through the angle $\varphi$, which transforms the XYZ axes into the X'Y'Z axes, where

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

is a counterclockwise rotation about the X'-axis through the angle $\theta$, which transforms theX'Y'Z axes into the X'Y"Z' axes, and where

$$R_z(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a counterclockwise rotation about the Z'-axis through the angle $\psi$, which transforms the X'Y"Z' axes into the X"Y"Z' axes.

**Definition 3.** *The angles $\varphi$, $\theta$, $\psi$ are called the* ***Euler angles*** *of the rotation.*

Thus, in terms of the Euler angle, an element $A = R_z(\psi)R_x(\theta)R_z(\varphi)$ of $SO(3)$ can be written as:

$$\begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix}$$

**4.2. $SU(2) = S^3$.** The Lie group $SU(2)$ can be identified with the group of all $2 \times 2$ unitary matrices, i.e., all matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

where $\alpha$ and $\beta$ are complex numbers such that

$$|\alpha|^2 + |\beta|^2 = 1,$$

and where $\alpha^*$ and $\beta^*$ denote the complex conjugates of $\alpha$ and $\beta$ repectively.

Letting $\alpha = x_1 + ix_2$ and $\beta = x_3 + ix_4$, we see that $SU(2)$ can be identified with

$$\left\{(x_1, x_2, x_3, x_4) \quad | \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\right\}$$

which is the standard 3-sphere $S^3$. Thus, $SU(2)$ can be identified with the standard 3-sphere $S^3$ of radius 1 in $\mathbb{R}^4$.

**4.3. $SU(2)$ as the universal cover of $SO(3)$.** We now construct a natural epimorphism

$$\Omega : SU(2) \longrightarrow SO(3)$$

as follows:

Identify $X = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with $M = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$. (Please note that $\{M\}$ is the set of traceless $2 \times 2$ skew hermitian matrices.) Then for every $Q \in SU(2)$, we have the map[1]

$$X = M \overset{\Lambda}{\longmapsto} QMQ^\dagger = M' = X'$$

We leave it to the reader to verify that this map preserves the standard norm on $\mathbb{R}^3$, i.e.,

$$|X| = |X'|$$

Hence, it follows that $\Lambda$ is an orthogonal transformation of $\mathbb{R}^3$ which we denote by $A = \Omega(Q)$. One can show that $\det(A) = 1$. Hence we have defined a map

$$\Omega : SU(2) \longrightarrow SO(3)$$

[1]Please note that, if $X$ is instead identified with $iM$, then $\Lambda$ is the adjoint map $Ad_Q : su(2) \longrightarrow su(2)$. Hence, $A = \Omega(Q) = Ad_Q$. Please also note that $M = X \cdot \overrightarrow{\sigma}$, where $\overrightarrow{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denotes the vector of pauli matrices. Thus, $\Lambda(X \cdot \overrightarrow{\sigma}) = QX \cdot \overrightarrow{\sigma} Q^\dagger = X' \cdot \overrightarrow{\sigma}$. Please refer to section 4.5.

which can be shown to be an epimorphism. This morphism is a 2-fold covering

$$S^3 \xrightarrow{\Omega} \mathbb{R}P^3$$

where

$$\Omega(Q) = \Omega(-Q)$$

In terms of the Euler angles we have

$$Q = Q_\psi Q_\theta Q_\varphi \longmapsto R_z(\psi) R_x(\theta) R_z(\varphi),$$

where

$$Q_\varphi = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \qquad Q_\theta = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & i\sin\left(\frac{\theta}{2}\right) \\ i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad Q_\psi = \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix},$$

and where

$$Q = \begin{pmatrix} e^{i(\psi+\varphi)/2}\cos\left(\frac{\theta}{2}\right) & ie^{i(\psi-\varphi)/2}\sin\left(\frac{\theta}{2}\right) \\ ie^{-i(\psi-\varphi)/2}\sin\left(\frac{\theta}{2}\right) & e^{-i(\psi+\varphi)/2}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

**Definition 4.** *The parameters $\alpha$, $\beta$, $\gamma$, $\delta$ in* $Q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU(2)$ *are called the **Cayley-Klein parameters** of $A \in SO(3)$.*

In terms of the Cayley-Klein parameters,

$$A = \Omega(Q) = \begin{pmatrix} \frac{1}{2}\left(\alpha^2 - \gamma^2 + \delta^2 - \beta^2\right) & \frac{i}{2}\left(\gamma^2 - \alpha^2 + \delta^2 - \beta^2\right) & \gamma\delta - \alpha\beta \\ -\frac{i}{2}\left(\alpha^2 + \gamma^2 - \beta^2 - \delta^2\right) & \frac{1}{2}\left(\alpha^2 + \gamma^2 + \beta^2 + \delta^2\right) & -i\left(\alpha\beta + \gamma\delta\right) \\ \beta\delta - \alpha\gamma & i\left(\alpha\gamma + \beta\delta\right) & \alpha\delta + \beta\gamma \end{pmatrix}$$

### 4.4. The Euler parameters.

Let

$$\begin{cases} \alpha = e_0 + ie_3 \\ \beta = e_2 + ie_1 \end{cases}$$

in the matrix $Q$ above. Then,

**Definition 5.** *The parameters $e_0$, $e_1$, $e_2$, $e_3$ are called the* ***Euler parameters*** *of $A$.*

Please note that

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

The Euler parameters are given below as functions of the Euler angles:

$$e_0 = \cos\left(\frac{\psi+\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right), \qquad e_2 = \sin\left(\frac{\psi-\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

$$e_1 = \cos\left(\frac{\psi-\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right), \qquad e_3 = \sin\left(\frac{\psi+\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

In terms of the Euler parameters,

$$A = \Omega(Q) = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2\left(e_1e_2 + e_0e_3\right) & 2\left(e_1e_3 - e_0e_2\right) \\ 2\left(e_1e_2 - e_0e_3\right) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2\left(e_2e_3 + e_0e_1\right) \\ 2\left(e_1e_3 + e_0e_2\right) & 2\left(e_2e_3 - e_0e_1\right) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}$$

### 4.5. The Pauli spin matrices.

The above formulas can be greatly simplified by using the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the identity matrix

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The Pauli spin matrices $\sigma_1$, $\sigma_2$, $\sigma_3$ satisfy the identities

$$\begin{cases} \sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}\mathbf{1} \\ [\sigma_j, \sigma_k] = 2i\sum_{\ell=1}^{3} \epsilon_{jk\ell}\sigma_\ell \\ \overline{\sigma}_j^T = \sigma_j \\ \sigma_j\sigma_k = i\sum_{\ell=1}^{3} \epsilon_{jk\ell}\sigma_\ell + \delta_{jk}\mathbf{1} \\ \left(\overrightarrow{a} \cdot \overrightarrow{\sigma}\right)\left(\overrightarrow{b} \cdot \overrightarrow{\sigma}\right) = \overrightarrow{a} \cdot \overrightarrow{b} + i\left(\overrightarrow{a} \times \overrightarrow{b}\right) \cdot \overrightarrow{\sigma} \end{cases}$$

where $\epsilon_{jk\ell}$ is the Levi-Civita tensor density defined by:

$$\epsilon_{jk\ell} = \begin{cases} 1 & \text{if } jk\ell \text{ is an even permutation of } 123 \\ -1 & \text{if } jk\ell \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise} \end{cases}$$

An element $Q$ of $SU(2)$ can now be written more simply in terms of the Euler parameters as

$$Q = e_0\mathbf{1} + i\left(e_1\sigma_1 + e_2\sigma_2 + e_3\sigma_3\right) = e_0\mathbf{1} + i\,\overrightarrow{e}\cdot\overrightarrow{\sigma}$$

Our identification,

$$X = (x_1, x_2, x_3) \longleftrightarrow M = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$$

can be more simply expressed as

$$X = (x_1, x_2, x_3) \longleftrightarrow M = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = X\cdot\overrightarrow{\sigma}$$

Moreover, the respective lifts $Q_\psi$, $Q_\theta$, $Q_\varphi$ to $SU(2)$ of the Euler rotations $R_z(\psi)$, $R_x(\theta)$, $R_z(\varphi)$ in $SO(3)$ can be more simply written as:

$$\begin{cases} Q_\psi = e^{i\sigma_1\psi/2} \\ Q_\varphi = e^{i\sigma_3\varphi/2} \\ Q_\theta = e^{i\sigma_1\theta/2} \end{cases}$$

Thus, a lift of the Euler angle decomposition is:

$$e^{i\sigma_1\psi/2}\cdot e^{i\sigma_3\varphi/2}\cdot e^{i\sigma_1\theta/2}$$

We will later see that the matrices $i\sigma_1$, $i\sigma_2$, $i\sigma_3$ are the infinitesimal generators of the Lie algebra $su(2)$ of $SU(2)$. Thus, $i\sigma_1$, $i\sigma_2$, $i\sigma_3$ can be respectively thought of as infintesimal rotations about the $x_1$, $x_2$, $x_3$ axes. The above formulas are an instance of the exponential map from $su(2)$ to $SU(2)$, which we will discussed later.

## 5. Computation of the Lie algebra $su(2)$ of $SU(2)$

$$SU(2) = \left\{ Q \in GL(2, \mathbb{C}) \mid Q\overline{Q}^T = I \text{ and } \det Q = 1 \right\}$$

The Lie algebra $su(2)$ consists of the tangent vectors $\dot{Q}(0)$ at $I$ of all curves $Q(t)$ in $SU(2)$ such that $Q(0) = I$.

Since $I = Q(t)\overline{Q(t)}^T$, it follows that

$$0 = \dot{Q}(t)\overline{Q(t)}^T + Q(t)\overline{\dot{Q}(t)}^T$$

But at $t = 0$, $Q(0) = I$ ; and we have

$$0 = \dot{Q}(0) + \overline{\dot{Q}(0)}^T .$$

Thus, all elements of $su(2)$ are skew hermitian matrices.

We consider the last condition, i.e.,

$$\det Q = 1$$

We can without loss of generality replace each curve $Q(t)$ with the curve

$$\widetilde{Q}(t) = e^{\dot{Q}(0)t},$$

since both curves have the same tangent vector at $t = 0$, i.e., at $I$. For the curve $\widetilde{Q}(t)$, we have

$$1 = \det \widetilde{Q}(t) = \det e^{\dot{Q}(0)t} = e^{Tr\left(\dot{Q}(0)t\right)}.$$

Hence, by taking the derivative, we have

$$0 = Tr\left(\dot{Q}(0)\right) e^{Tr\left(\dot{Q}(0)t\right)}.$$

Thus, at $t = 0$, we have

$$Tr\left(\dot{Q}(0)\right) = 0.$$

Thus, $su(2)$ consists of all traceless skew hermitian matrices[2].

[2]The traceless condition for the elements of $su(2)$ could also have been proven directly with the use of the matrix identity $\partial \det A / \partial A = \left(A^T\right)^{-1} \det A$.

## 6. Computation of the Lie algebra $so(3)$ of $SO(3)$.

$$SO(3) = \left\{ A \in GL(3,\mathbb{R}) \mid AA^T = I \text{ and } \det A = 1 \right\}$$

Thus, the Lie algebra $so(3)$ consists of thae tangent vectors $\dot{A}(0)$ at $I$ of all curves $A(t)$ in $SO(3)$ such that $A(0) = I$.

Since

$$I = A(t)A(t)^T,$$

it follows that

$$0 = \dot{A}(t)A(t)^T + A(t)\dot{A}(t)^T.$$

But at $t = 0$, $A(0) = I$. Thus,

$$0 = \dot{A}(0) + \dot{A}(0)^T.$$

Thus each such tangent vector is skew symmetric. Hence, $so(3)$ consists of all skew symmetric matrices.

**Remark 2.** *The condition* $\det A = 1$ *need not be considered because it is implied by the facts that* $I \in SO(3)$, $AA^T = 1$, *and* $SO(3)$ *is arcwise connected. For the condition* $AA^T = I$ *implies that* $\det A = \pm 1$. *But* $I \in SO(3)$ *and the fact that* $SO(3)$ *is arcwise connected imply by continuity of* $\det$ *that all elements of* $SO(3)$ *have the same determinant as* $I$. *Even if the condition* $\det(A) = 1$ *is considered, it will imply that* $tr(\dot{A}(0)) = 0$. *But all skew symmetric matrices are traceless anyway.*

## 7. The adjoint representations $Ad$ and $ad$ of a Lie group.

Let $G$ be a Lie group, and $g$ its Lie algebra. For every $Q$ in $G$, we define an inner automorphism

$$I_Q : G \longrightarrow G$$

by

$$M \longmapsto I_Q(M) = QMQ^{-1}.$$

As a result, we have the following commutative diagram

$$\begin{array}{ccc} TG & \xrightarrow{dI_Q} & TG \\ \pi \downarrow & & \downarrow \pi \\ G & \xrightarrow{I_Q} & G \end{array},$$

where $TG$ denotes the tangent bundle of $G$. We define

$$Ad : G \longrightarrow Aut(g)$$

by

$$Ad_Q = (dI_Q)_e \ ,$$

where $e$ denotes the idenity of $G$, where $Aut(g)$ denotes the group of automorphisms of the Lie algebra $g$, and where $(dI_Q)_e$ denotes the restriction of $dI_Q$ to the fibre $\pi^{-1}(e)$.

Let $End(g)$ denote the ring of endomorphisms of the Lie algebra $g$. Then the adjoint representation

$$ad : g \longrightarrow End(g)$$

is defined as

$$ad_X(Y) = [X, Y] \, ,$$

for all $X$, $Y \in g$.

It can be shown that the Lie algebra of the Lie group $Aut(g)$ is $End(g)$, and that the following diagram

$$\begin{array}{ccc} g & \xrightarrow{d(Ad)_e} & End(g) \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{Ad} & Aut(g) \end{array}$$

is commutative, and that

$$ad = d\,(Ad)_e$$

## 8. THE ADJOINT REPRESENTATION $Ad : SU(2) \longrightarrow Aut(su(2))$

From the previous section, we know that, for each $Q \in SU(2)$,

$$Ad_Q = d\,(I_Q)_I \ .$$

**Proposition 2.** *For all $Q$ in $SU(2)$ and for all $B$ in $su(2)$,*

$$Ad_Q(B) = I_Q B = QBQ^{-1}$$

**Proof:**

By definition, $d\left(I_Q\right)_I$ takes the tangent vector to the curve

$$e^{Bt}$$

at $t = 0$ to the tangent vector to the curve

$$I_Q e^{Bt}$$

at $t = 0$. But the tangent vector to $I_Q e^{Bt}$ at $t = 0$ is given by

$$d\left(I_Q\right)_I B = \lim_{t\to 0}\left(\frac{I_Q e^{Bt} - I_Q e^{B\cdot 0}}{t}\right) = \lim_{t\to 0}\left(\frac{Q\left(e^{Bt} - e^{B\cdot 0}\right)Q^{-1}}{t}\right) = Q\lim_{t\to 0}\left(\frac{e^{Bt} - e^{B\cdot 0}}{t}\right)Q^{-1},$$

where the last statement follows from the continuity of matrix multiplication.

Therefore,

$$d\left(I_Q\right)_I B = I_Q B$$

**Q.E.D.**

**Remark 3.** *Moreover, since*

$$I_Q e^{Bt} = Qe^{Bt}Q^{-1} = e^{QBQ^{-1}t} = e^{I_Q Bt},$$

*we have the following diagram*

$$\begin{array}{ccc} su(2) & \xrightarrow{Ad_Q} & su(2) \\ \exp\downarrow & & \downarrow\exp \\ SU(2) & \xrightarrow{I_Q} & SU(2) \end{array}$$

We now compute the adjoint representation $Ad$ of $SU(2)$ in terms of the basis

$$E_1 = -i\sigma_1,\ E_2 = -i\sigma_2,\ E_3 = -i\sigma_3$$

where $\sigma_1$, $\sigma_2$, $\sigma_3$ are the Pauli spin matrices defined on page 9 of these notes.

**Remark 4.** *Please note that the Pauli spin matrices do not form a basis of* $su(2)$*. In fact, they are not even elements of* $su(2)$*.*

Let $Q$ be an arbitray element of $SU(2)$. Hence,

$$Q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} x_0 + ix_3 & x_2 + ix_1 \\ -x_2 + ix_1 & x_0 - ix_3 \end{pmatrix},$$

where

$$|\alpha|^2 + |\beta|^2 = 1 = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

We now compute

$$Ad_Q(E_1)\,,\ Ad_Q(E_2)\,,\ Ad_Q(E_3)$$

$$\begin{aligned} Ad_Q(E_1) &= \begin{pmatrix} x_0 + ix_3 & x_2 + ix_1 \\ -x_2 + ix_1 & x_0 - ix_3 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 - ix_2 & x_3 - ix_0 \\ -x_3 - ix_0 & x_1 + ix_2 \end{pmatrix} \begin{pmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{pmatrix} \\ &= \begin{pmatrix} (x_1 - ix_2)(x_0 - ix_3) + (x_3 - ix_0)(x_2 - ix_1) & (x_1 - ix_2)(-x_2 - ix_1) + (x_3 - ix_0)(x_0 + ix_3) \\ (-x_3 - ix_0)(x_0 - ix_3) + (x_1 + ix_2)(x_2 - ix_1) & (-x_3 - ix_0)(-x_2 - ix_1) + (x_1 + ix_2)(x_0 + ix_3) \end{pmatrix} \end{aligned}$$

Therefore,

$$Ad_Q E_1 = \begin{pmatrix} -2ix_0x_2 - 2ix_1x_3 & 2x_0x_3 - 2x_1x_2 + i\left(-x_0^2 - x_1^2 + x_2^2 + x_3^2\right) \\ -2x_0x_3 + 2x_1x_2 + i\left(-x_0^2 - x_1^2 + x_2^2 + x_3^2\right) & 2ix_0x_2 + 2ix_1x_3 \end{pmatrix}$$

Thus,

$$Ad_Q E_1 = (2x_0x_2 + 2x_1x_3)\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + (2x_0x_3 - 2x_1x_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \left(-x_0^2 - x_1^2 + x_2^2 + x_3^2\right)\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

But

$$E_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

So we finally have:

$$Ad_Q E_1 = \left(x_0^2 + x_1^2 - x_2^2 - x_3^2\right) E_1 + (2x_1x_2 - 2x_0x_3) E_2 + (2x_0x_2 + 2x_1x_3) E_3$$

Please note that we have computed the first column of the matrix $A = \Omega(Q)$ given on page 9 of these notes. Thes the $x_j$'s are the same as the Euler parameters $e_j$'s. We leave the computation of $Ad_Q E_2$ and $Ad_Q E_3$ as an exercise for the reader[3].

## 9. The adjoint representation $ad : su(2) \longrightarrow End\,(su(2))$

Again, we use the basis

$$E_j = -i\sigma_j$$

for our calculations.

$$ad_{E_j}(E_k) = [E_j, E_k] = -[\sigma_j, \sigma_k] = 2i\sum_{\ell=1}^{3} \epsilon_{jk\ell}\sigma_\ell,$$

where we have used the identity found on page 9 of these notes. Therefore, we have

$$ad_{E_j}(E_k) = -2\sum_{\ell=1}^{3} \epsilon_{jk\ell}E_\ell$$

[3]It should be possible to greatly simplify the calculation of $Ad$ for $SO(3)$ by using the Pauli spin matrix identies found on page 9 of these notes.

Hence, by the bilinearity of the Lie bracket,

$$\begin{aligned} ad_{\sum_j a_j E_j}\left(\sum_k b_k E_k\right) &= \sum_j \sum_k a_j b_k \left(-2\sum_\ell \epsilon_{jk\ell} E_\ell\right) \\ &= \sum_\ell \left(\sum_{j,k} -2 a_j b_k \epsilon_{jk\ell}\right) E_\ell \\ &= \sum_\ell -2\left(\overrightarrow{a} \times \overrightarrow{b}\right)_\ell E_\ell \end{aligned}$$

Thus,

$$ad_{\overrightarrow{a}}\left(\overrightarrow{b}\right) = -2\overrightarrow{a} \times \overrightarrow{b}$$

where $\overrightarrow{a} \times \overrightarrow{b}$ denotes the vector cross product.

But,

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

So finally,

$$ad_{\overrightarrow{a}} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

The killing form

$$B(\overrightarrow{a}, \overrightarrow{b}) = Tr\left(ad_{\overrightarrow{a}}\, ad_{\overrightarrow{b}}\right)$$

is

$$\begin{aligned} B(\overrightarrow{a}, \overrightarrow{b}) &= Tr\left(\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}\right) \\ &= Tr\begin{pmatrix} -a_2b_2 - a_3b_3 & a_2b_1 & a_3b_1 \\ a_1b_2 & -a_1b_1 - a_3b_3 & a_3b_2 \\ a_1b_3 & a_2b_3 & -a_1b_1 - a_2b_2 \end{pmatrix} \\ &= -2\overrightarrow{a} \cdot \overrightarrow{b} \end{aligned}$$

## 10. THE ADJOINT REPRESENTATION OF $Ad : SO(3) \longrightarrow Aut(so(3))$

As with $SU(2)$, we have:

**Proposition 3.** *For all $A \in SO(3)$ and for all $B \in so(3)$, we have*

$$Ad_A B = I_A B = ABA^{-1}$$

*Moreover, the diagram*

$$\begin{array}{ccc} so(3) & \xrightarrow{Ad_A} & so(3) \\ \exp \downarrow & & \downarrow \exp \\ SO(3) & \xrightarrow{I_A} & SO(3) \end{array}$$

*is commutative.*

We compute the adjoint representation in terms of the following basis of $so(3)$

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We leave it for the reader to verify the following identity,

$$[L_j, L_k] = \sum_\ell \epsilon_{jk\ell} L_\ell$$

Let $A$ be an arbitrary element of $SO(3)$. Then $A$ is of the form

$$A = e^{\sum_j \theta_j L_j}$$

Thus,

$$Ad_A L_j = e^{\sum_j \theta_j L_j} L_j e^{-\sum_j \theta_j L_j}$$

We now need to rewrite the above expression as a linear combination of the basis elements $\{L_j\}$.

Ugh!

## 11. THE ADJOINT REPRESENTATION $ad : so(3) \longrightarrow End\,(so(3))$

Again, we use the basis $\{L_j\}$ for our calculation.

$$\begin{aligned} ad_{\sum_j \theta_j L_j}\left(\sum_j \varphi_j L_j\right) &= \sum_{j,k} \theta_j \varphi_k ad_{L_j}\left(L_k\right) = \sum_{j,k} \theta_j \varphi_k \left[L_j, L_k\right] \\ &= \sum_{\ell} \sum_{j,k} \theta_j \varphi_k \epsilon_{jk\ell} L_\ell = \sum_{\ell} \left(\overrightarrow{\theta} \times \overrightarrow{\varphi}\right)_\ell L_\ell \end{aligned}$$

Hence,

$$ad_{\overrightarrow{\theta}}\left(\overrightarrow{\varphi}\right) = \overrightarrow{\theta} \times \overrightarrow{\varphi}$$

So finally,

$$ad_{\overrightarrow{\theta}} = \begin{pmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{pmatrix}$$

It also follows that the killing form is

$$B\left(\overrightarrow{\theta}, \overrightarrow{\varphi}\right) = -2\overrightarrow{\theta} \cdot \overrightarrow{\varphi}$$

## REFERENCES

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[3] Pontrjagin, L., **Topological Groups,** Princeton University Press, (1958).

[4] Sattinger, D.H., and O. L. Weaver, **Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics, Springer-Verlag (1993).**